MATHEMATICS – 9
PTB - SYLLABUS

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Unit 1 Real Numbers

EXERCISE 1.1

1. Identify each of the following as a rational or irrational number:

(i)	2.353535	(ii)	$0.\overline{6}$	(iii)	$2.236067\ldots$	(iv)	$\sqrt{7}$
(v)	e	(vi)	π	(vii)	$5+\sqrt{11}$	(viii)	$\sqrt{3}+\sqrt{13}$
(ix)	$\frac{15}{4}$	(x)	$(2-\sqrt{2})(2+\sqrt{2})$				

Solution

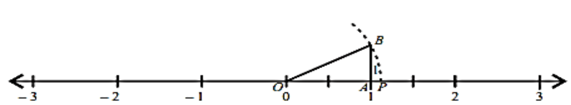
(i) Rational	(ii) Rational	(iii) Irrational	(iv) Irrational	(v) Irrational
(vi) Irrational	(vii) Irrational	(viii) Irrational	(ix) Rational	(x) rational

2. Represent the following numbers on number line:

(i)	$\sqrt{2}$	(ii)	$\sqrt{3}$	(iii)	$4\frac{1}{3}$
(iv)	$-2\frac{1}{7}$	(v)	$\frac{5}{8}$	(vi)	$2\frac{3}{4}$

Solution

(i)



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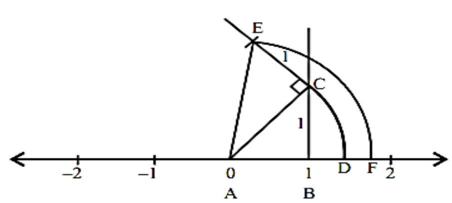

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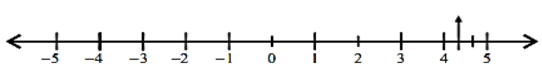

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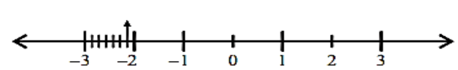

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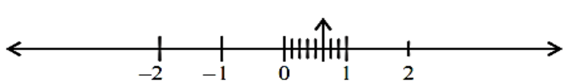

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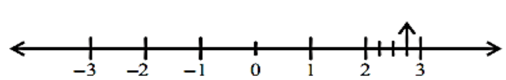


3. Express the following as a rational number $\frac{p}{q}$ where p and q are integers and $q \neq 0$:

(i) $0.\overline{4}$ (ii) $0.\overline{37}$ (iii) $0.\overline{21}$

$\boldsymbol{x = 0.\overline{4}}$ $x = 0.4444\ldots$ $10x = 10(0.4444\ldots)$ $10x = 4.4444\ldots$ $10x - x = (4.4444\ldots) - (0.4444\ldots)$ $9x = 4 \Rightarrow \boldsymbol{x = \frac{4}{9}}$	$\boldsymbol{x = 0.\overline{37}}$ $x = 0.3737\ldots$ $100x = 100(0.3737\ldots)$ $100x = 37.3737\ldots$ $100x - x = (37.3737\ldots) - (0.3737\ldots)$ $99x = 37 \Rightarrow \boldsymbol{x = \frac{37}{99}}$

$$\boldsymbol{x = 0.\overline{21}}$$

$$x = 0.2121\ldots$$

$$100x = 100(0.2121\ldots)$$

$$100x = 21.2121\ldots$$

$$100x - x = (21.2121\ldots) - (0.2121\ldots)$$

$$99x = 21 \Rightarrow \boldsymbol{x = \frac{21}{99}}$$

4. Name the property used in the following:

(i) $(a+4)+b = a+(4+b)$ (ii) $\sqrt{2}+\sqrt{3} = \sqrt{3}+\sqrt{2}$

(iii) $x - x = 0$ (iv) $a(b+c) = ab+ac$

(v) $16+0 = 16$ (vi) $100 \times 1 = 100$

(vii) $4 \times (5 \times 8) = (4 \times 5) \times 8$ (viii) $ab = ba$

Solution

(i) Associative property over addition
(ii) Commutative property over addition
(iii) Additive inverse
(iv) Left distributive property
(v) Additive identity
(vi) Multiplicative identity
(vii) Associative property under multiplication
(viii) Commutative property under multiplication

5. Name the property used in the following:

(i) $-3 < -1 \Rightarrow 0 < 2$
(ii) If $a < b$ then $\frac{1}{a} > \frac{1}{b}$
(iii) If $a < b$ then $a + c < b + c$
(iv) If $ac < bc$ and $c > 0$ then $a < b$
(v) If $ac < bc$ and $c < 0$ then $a > b$
(vi) Either $a > b$ or $a = b$ or $a < b$

Solution

(i) Additive property
(ii) Reciprocal property
(iii) Additive property
(iv) Multiplicative property
(v) Multiplicative property
(vi) Trichotomy property

6. Insert two rational numbers between:

(i) $\frac{1}{3}$ and $\frac{1}{4}$
(ii) 3 and 4
(iii) $\frac{3}{5}$ and $\frac{4}{5}$

Solution

i. $q_1 = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\right) = \frac{1}{2}\left(\frac{7}{12}\right) = \frac{7}{24}$ and $q_2 = \frac{1}{2}\left(\frac{7}{24} + \frac{1}{4}\right) = \frac{1}{2}\left(\frac{13}{24}\right) = \frac{13}{48}$

Hence required rational are $\mathbf{\frac{7}{24}, \frac{13}{48}}$

ii. $q_1 = \frac{1}{2}(3 + 4) = \frac{7}{2}$ and $q_2 = \frac{1}{2}\left(\frac{7}{2} + 4\right) = \frac{1}{2}\left(\frac{15}{2}\right) = \frac{15}{4}$

Hence required rational are $\mathbf{\frac{7}{2}, \frac{15}{4}}$

iii. $q_1 = \frac{1}{2}\left(\frac{3}{5} + \frac{4}{5}\right) = \frac{1}{2}\left(\frac{7}{5}\right) = \frac{7}{10}$ and $q_2 = \frac{1}{2}\left(\frac{7}{10} + \frac{4}{5}\right) = \frac{1}{2}\left(\frac{15}{10}\right) = \frac{3}{4}$

Hence required rational are $\mathbf{\frac{7}{10}, \frac{3}{4}}$

EXERCISE 1.2

1. Rationalize the denominator of following:

(i) $\dfrac{13}{4+\sqrt{3}}$ (ii) $\dfrac{\sqrt{2}+\sqrt{5}}{\sqrt{3}}$ (iii) $\dfrac{\sqrt{2}-1}{\sqrt{5}}$

(iv) $\dfrac{6-4\sqrt{2}}{6+4\sqrt{2}}$ (v) $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ (vi) $\dfrac{4\sqrt{3}}{\sqrt{7}+\sqrt{5}}$

Solution

i. $\dfrac{\mathbf{13}}{\mathbf{4+\sqrt{3}}} = \dfrac{13}{4+\sqrt{3}} \times \dfrac{4-\sqrt{3}}{4-\sqrt{3}} = \dfrac{13(4-\sqrt{3})}{(4)^2-(\sqrt{3})^2} = \dfrac{13(4-\sqrt{3})}{16-3} = \dfrac{13(4-\sqrt{3})}{13} = \mathbf{4-\sqrt{3}}$

ii. $\dfrac{\mathbf{\sqrt{2}+\sqrt{5}}}{\mathbf{\sqrt{3}}} = \dfrac{\sqrt{2}+\sqrt{5}}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{(\sqrt{2}+\sqrt{5})\sqrt{3}}{\sqrt{3}.\sqrt{3}} = \dfrac{\sqrt{2}.\sqrt{3}+\sqrt{5}.\sqrt{3}}{3} = \dfrac{\mathbf{\sqrt{6}+\sqrt{15}}}{\mathbf{3}}$

iii. $\dfrac{\mathbf{\sqrt{2}-1}}{\mathbf{\sqrt{5}}} = \dfrac{\sqrt{2}-1}{\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{(\sqrt{2}-1)\sqrt{5}}{\sqrt{5}.\sqrt{5}} = \dfrac{\sqrt{2}.\sqrt{5}-1.\sqrt{5}}{5} = \dfrac{\mathbf{\sqrt{10}-\sqrt{5}}}{\mathbf{5}}$

iv. $\dfrac{\mathbf{6-4\sqrt{2}}}{\mathbf{6+4\sqrt{2}}} = \dfrac{6-4\sqrt{2}}{6+4\sqrt{2}} \times \dfrac{6-4\sqrt{2}}{6-4\sqrt{2}} = \dfrac{(6-4\sqrt{2})^2}{(6)^2-(4\sqrt{2})^2} = \dfrac{(6)^2+(4\sqrt{2})^2-2(6)(4\sqrt{2})}{36-16(2)}$

$= \dfrac{36+32-48\sqrt{2}}{36-32} = \dfrac{68-48\sqrt{2}}{4} = \mathbf{17-12\sqrt{2}}$

v. $\dfrac{\mathbf{\sqrt{3}-\sqrt{2}}}{\mathbf{\sqrt{3}+\sqrt{2}}} = \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} \times \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \dfrac{(\sqrt{3}-\sqrt{2})^2}{(\sqrt{3})^2-(\sqrt{2})^2} = \dfrac{(\sqrt{3})^2+(\sqrt{2})^2-2(\sqrt{3})(\sqrt{2})}{3-2}$

$= \dfrac{3+2-2\sqrt{6}}{1} = \mathbf{5-2\sqrt{6}}$

vi. $\dfrac{\mathbf{4\sqrt{3}}}{\mathbf{\sqrt{7}+\sqrt{5}}} = \dfrac{4\sqrt{3}}{\sqrt{7}+\sqrt{5}} \times \dfrac{\sqrt{7}-\sqrt{5}}{\sqrt{7}-\sqrt{5}} = \dfrac{4\sqrt{3}(\sqrt{7}-\sqrt{5})}{(\sqrt{7})^2-(\sqrt{5})^2} = \dfrac{4\sqrt{3}(\sqrt{7}-\sqrt{5})}{7-5}$

$= \dfrac{4\sqrt{3}(\sqrt{7}-\sqrt{5})}{2} = \mathbf{2\sqrt{3}}(\sqrt{7}-\sqrt{5})$

2. Simplify the following:

(i) $\left(\frac{81}{16}\right)^{-\frac{3}{4}}$ (ii) $\left(\frac{3}{4}\right)^{-2} \div \left(\frac{4}{9}\right)^{3} \times \frac{16}{27}$ (iii) $(0.027)^{-\frac{1}{3}}$

(iv) $\sqrt[7]{\frac{x^{14} \times y^{21} \times z^{35}}{y^{14} z^{7}}}$ (v) $\frac{5\cdot(25)^{n+1} - 25\cdot(5)^{2n}}{5\cdot(5)^{2n+3} - (25)^{n+1}}$

(vi) $\frac{(16)^{x+1} + 20(4^{2x})}{2^{x-3} \times 8^{x+2}}$ (vii) $(64)^{-\frac{2}{3}} \div (9)^{-\frac{3}{2}}$

(viii) $\frac{3^{n} \times 9^{n+1}}{3^{n-1} \times 9^{n-1}}$ (ix) $\frac{5^{n+3} - 6.5^{n+1}}{9 \times 5^{n} - 2^{n} \times 5^{n}}$

Solution

i. $\left(\frac{81}{16}\right)^{-\frac{3}{4}} = \left(\frac{16}{81}\right)^{\frac{3}{4}} = \left(\frac{2^4}{3^4}\right)^{\frac{3}{4}} = \frac{2^{4\times\frac{3}{4}}}{3^{4\times\frac{3}{4}}} = \frac{2^3}{3^3} = \mathbf{\frac{8}{27}}$

ii. $\left(\frac{3}{4}\right)^{-2} \div \left(\frac{4}{9}\right)^{3} \times \frac{16}{27} = \left(\frac{4}{3}\right)^{2} \div \left(\frac{4}{9}\right)^{3} \times \frac{16}{27} = \frac{4^2}{3^2} \times \frac{9^3}{4^3} \times \frac{16}{27} = \frac{16\times729\times16}{9\times64\times27} = \mathbf{12}$

iii. $(0.027)^{-\frac{1}{3}} = \left(\frac{27}{1000}\right)^{-\frac{1}{3}} = \left(\frac{1000}{27}\right)^{\frac{1}{3}} = \left(\frac{10^3}{3^3}\right)^{\frac{1}{3}} = \frac{10^{3\times\frac{1}{3}}}{3^{3\times\frac{1}{3}}} = \mathbf{\frac{10}{3}}$

iv. $\sqrt[7]{\frac{x^{14}\times y^{21}\times z^{35}}{y^{14}\times z^{7}}} = \left(\frac{x^{14}\times y^{21}\times z^{35}}{y^{14}\times z^{7}}\right)^{\frac{1}{7}} = (x^{14} \times y^{7} \times z^{28})^{\frac{1}{7}}$

$= x^{14\times\frac{1}{7}} \times y^{7\times\frac{1}{7}} \times z^{28\times\frac{1}{7}} = \boldsymbol{x^2yz^4}$

v. $\frac{5.(25)^{n+1}-25.(5)^{2n}}{5.(5)^{2n+3}-(25)^{n+1}} = \frac{5.(5^2)^{n+1}-5^2.(5)^{2n}}{5.(5)^{2n+3}-(5^2)^{n+1}} = \frac{5.5^{2n+2}-5^2.5^{2n}}{5.5^{2n+3}-5^{2n+2}} = \frac{5^{2n+3}-5^{2n+2}}{5^{2n+4}-5^{2n+2}}$

$= \frac{5^{2n+2}(5-1)}{5^{2n+2}(5^2-1)} = \frac{5-1}{25-1} = \frac{4}{24} = \mathbf{\frac{1}{6}}$

vi. $\frac{(16)^{x+1}+20(4^{2x})}{2^{x-3}\times8^{x+2}} = \frac{(2^4)^{x+1}+20.(2^2)^{2x}}{2^{x-3}\times(2^3)^{x+2}} = \frac{2^{4x+4}+20.2^{4x}}{2^{x-3}\times2^{3x+6}} = \frac{2^{4x+4}+20.2^{4x}}{2^{4x+3}}$

$= \frac{2^{4x}(2^4+20)}{2^{4x}.2^3} = \frac{(16+20)}{2^3} = \frac{36}{8} = \mathbf{\frac{9}{2}}$

vii. $(\mathbf{64})^{-\frac{2}{3}} \div (\mathbf{9})^{-\frac{3}{2}} = \frac{(64)^{-\frac{2}{3}}}{(9)^{-\frac{3}{2}}} = \frac{(9)^{\frac{3}{2}}}{(64)^{\frac{2}{3}}} = \frac{(3^2)^{\frac{3}{2}}}{(4^3)^{\frac{2}{3}}} = \frac{3^3}{4^2} = \mathbf{\frac{27}{16}}$

viii. $\frac{\mathbf{3^n \times 9^{n+1}}}{\mathbf{3^{n-1} \times 9^{n-1}}} = \frac{3^n \times (3^2)^{n+1}}{3^{n-1} \times (3^2)^{n-1}} = \frac{3^n \times 3^{2n+2}}{3^{n-1} \times 3^{2n-2}} = \frac{3^{3n+2}}{3^{3n-3}} = 3^{3n+2-3n+3} = 3^5 = \mathbf{243}$

ix. $\frac{\mathbf{5^{n+3} - 6.5^{n+1}}}{\mathbf{9 \times 5^n - 2^n \times 5^n}}$ =???

$\frac{\mathbf{5^{n+3}-6.5^{n+1}}}{\mathbf{9\times5^n-2^n\times5^n}}$ **wrong statement**	$\frac{\mathbf{5^{n+3}-6.5^{n+1}}}{\mathbf{9\times5^n-2^n\times5^n}}$ **according to book**
$\frac{\mathbf{5^{n+3}-6.5^{n+1}}}{\mathbf{9\times5^n-2^2\times5^n}}$ **right statement**	$= \frac{5^{n+1}(5^2-6)}{5^n(9-2^n)} = \frac{5(5^2-6)}{(9-2^n)}$
$= \frac{5^n(5^3-6.5^1)}{5^n(9-2^2)}$	$= \frac{5(25-6)}{(9-2^n)} = \frac{5(19)}{(9-2^n)} = \frac{5(19)}{(9-2^2)}$; $n = 2$
$= \frac{125-30}{9-4} = \frac{95}{5} = \mathbf{19}$	$= \frac{5(19)}{9-4} = \frac{5(19)}{5} = \mathbf{19}$

3. If $x = 3 + \sqrt{8}$ then find the value of:

(i) $x + \frac{1}{x}$ (ii) $x - \frac{1}{x}$ (iii) $x^2 + \frac{1}{x^2}$

(iv) $x^2 - \frac{1}{x^2}$ (v) $x^4 + \frac{1}{x^4}$ (vi) $\left(x - \frac{1}{x}\right)^2$

Solution

$x = 3 + \sqrt{8} \Rightarrow \frac{1}{x} = \frac{1}{3+\sqrt{8}} = \frac{1}{3+\sqrt{8}} \times \frac{3-\sqrt{8}}{3-\sqrt{8}} = \frac{3-\sqrt{8}}{(3)^2-(\sqrt{8})^2} = \frac{3-\sqrt{8}}{9-8} = 3 - \sqrt{8}$

Hence $\boldsymbol{x} = \mathbf{3} + \sqrt{\mathbf{8}}$ and $\frac{\mathbf{1}}{\boldsymbol{x}} = \mathbf{3} - \sqrt{\mathbf{8}}$

i. $x + \frac{1}{x} = (3 + \sqrt{8}) + (3 - \sqrt{8}) = \mathbf{6}$

ii. $x - \frac{1}{x} = (3 + \sqrt{8}) - (3 - \sqrt{8}) = \mathbf{2\sqrt{8}}$

iii. $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = (6)^2 - 2 = 36 - 2 = \mathbf{34}$

iv. $x^2 - \frac{1}{x^2} = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right) = (6)\left(2\sqrt{8}\right) = \mathbf{12\sqrt{8}}$

v. $x^4 + \frac{1}{x^4} = \left(x^2 + \frac{1}{x^2}\right)^2 - 2 = (34)^2 - 2 = 1156 - 2 = \mathbf{1154}$

vi. $\left(x - \frac{1}{x}\right)^2 = \left(2\sqrt{8}\right)^2 = 4 \times 8 = \mathbf{32}$

4. Find the rational numbers p and q such that $\dfrac{8-3\sqrt{2}}{4+3\sqrt{2}} = p + q\sqrt{2}$

Solution

$\frac{8-3\sqrt{2}}{4+3\sqrt{2}} = p + q\sqrt{2}$

$\frac{8-3\sqrt{2}}{4+3\sqrt{2}} \times \frac{4-3\sqrt{2}}{4-3\sqrt{2}} = p + q\sqrt{2}$

$\frac{32-24\sqrt{2}-12\sqrt{2}+18}{(4)^2-\left(3\sqrt{2}\right)^2} = p + q\sqrt{2}$

$\frac{50-36\sqrt{2}}{16-18} = p + q\sqrt{2}$

$\frac{50-36\sqrt{2}}{-2} = p + q\sqrt{2}$

$-25 + 18\sqrt{2} = p + q\sqrt{2}$

Hence $\boldsymbol{p} = \mathbf{-25}$ and $\boldsymbol{q} = \mathbf{18}$

5. Simplify the following:

(i) $\dfrac{(25)^{\frac{3}{2}} \times (243)^{\frac{3}{5}}}{(16)^{\frac{5}{4}} \times (8)^{\frac{4}{3}}}$ (ii) $\dfrac{54 \times \sqrt[3]{(27)^{2x}}}{9^{x+1} + 216(3^{2x-1})}$

(iii) $\sqrt{\dfrac{(216)^{\frac{2}{3}} \times (25)^{\frac{1}{2}}}{(0.04)^{\frac{-3}{2}}}}$ (iv) $\left(a^{\frac{1}{3}} + b^{\frac{2}{3}}\right) \times \left(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}\right)$

Solution

i. $\dfrac{\boldsymbol{(25)^{\frac{3}{2}}\times(243)^{\frac{3}{5}}}}{\boldsymbol{(16)^{\frac{5}{4}}\times(8)^{\frac{4}{3}}}}=\dfrac{(5^2)^{\frac{3}{2}}\times(3^5)^{\frac{3}{5}}}{(2^4)^{\frac{5}{4}}\times(2^3)^{\frac{4}{3}}}=\dfrac{5^3\times3^3}{2^5\times2^4}=\dfrac{5^3\times3^3}{2^9}=\dfrac{125\times27}{512}=\boldsymbol{\dfrac{3375}{512}}$

ii. $\dfrac{\boldsymbol{54\times\sqrt[3]{(27)^{2x}}}}{\boldsymbol{9^{x+1}+216(3^{2x-1})}}=\dfrac{54\times(27)^{\frac{2x}{3}}}{9^{x+1}+216(3^{2x-1})}=\dfrac{54\times(3^3)^{\frac{2x}{3}}}{(3^2)^{x+1}+216(3^{2x-1})}=\dfrac{54\times3^{2x}}{3^{2x+2}+216(3^{2x-1})}$

$$=\frac{54\times3^{2x}}{3^{2x}\left(3^2+216(3^{-1})\right)}=\frac{54}{\left(3^2+\frac{216}{3}\right)}=\frac{54}{9+72}=\frac{54}{81}=\boldsymbol{\frac{2}{3}}$$

iii. $\sqrt{\dfrac{\boldsymbol{(216)^{\frac{2}{3}}\times(25)^{\frac{1}{2}}}}{\boldsymbol{(0.04)^{-\frac{3}{2}}}}}=\left(\dfrac{(6^3)^{\frac{2}{3}}\times(5^2)^{\frac{1}{2}}}{\left(\frac{4}{100}\right)^{-\frac{3}{2}}}\right)^{\frac{1}{2}}=\left(\dfrac{6^2\times5}{\left(\frac{100}{4}\right)^{\frac{3}{2}}}\right)^{\frac{1}{2}}=\left(\dfrac{6^2\times5}{(25)^{\frac{3}{2}}}\right)^{\frac{1}{2}}=\left(\dfrac{6^2\times5}{(5^2)^{\frac{3}{2}}}\right)^{\frac{1}{2}}$

$$=\left(\frac{6^2\times5}{5^3}\right)^{\frac{1}{2}}=\left(\frac{6^2}{5^2}\right)^{\frac{1}{2}}=\boldsymbol{\frac{6}{5}}$$

iv. $\boldsymbol{\left(a^{\frac{1}{3}}+b^{\frac{2}{3}}\right)\times\left(a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}\right)}$

$$=\left(a^{\frac{1}{3}}+b^{\frac{2}{3}}\right)\times\left(a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}\right)$$

$$=\left(a^{\frac{1}{3}}a^{\frac{2}{3}}-a^{\frac{1}{3}}a^{\frac{1}{3}}b^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{4}{3}}+b^{\frac{2}{3}}a^{\frac{2}{3}}-b^{\frac{2}{3}}a^{\frac{1}{3}}b^{\frac{2}{3}}+b^{\frac{2}{3}}b^{\frac{4}{3}}\right)$$

$$=\left(a^{\frac{1}{3}+\frac{2}{3}}-a^{\frac{1}{3}+\frac{1}{3}}b^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{4}{3}}+a^{\frac{2}{3}}b^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{2}{3}+\frac{2}{3}}+b^{\frac{2}{3}+\frac{4}{3}}\right)$$

$$=\left(a^{\frac{3}{3}}-\cancel{a^{\frac{2}{3}}b^{\frac{2}{3}}}+\cancel{a^{\frac{1}{3}}b^{\frac{4}{3}}}+\cancel{a^{\frac{2}{3}}b^{\frac{2}{3}}}-\cancel{a^{\frac{1}{3}}b^{\frac{4}{3}}}+b^{\frac{6}{3}}\right)$$

$$=\boldsymbol{a+b^2}$$

EXERCISE 1.3

1. The sum of three consecutive integers is forty-two, find the three integers.

Solution

Consider three consecutive integers are x , $(x+1)$ and $(x+2)$

$(x)+(x+1)+(x+2)=42$

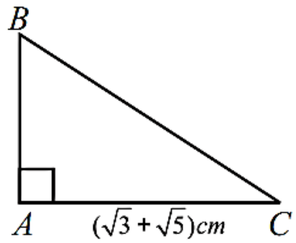
$3x+3=42$

$3x=39$

$x=13$

Hence the three consecutive integers are **13**, **14**, and **15**.

2. The diagram shows right angled ΔABC in which the length of $\overline{AC}$ is $(\sqrt{3}+\sqrt{5})$cm. The area of ΔABC is $(1+\sqrt{15})$ cm². Find the length $\overline{AB}$ in the form $(a\sqrt{3}+b\sqrt{5})$ cm, where a and b are integers.



Solution

Length of $\overline{AC}=(\sqrt{3}+\sqrt{5})$ cm

Area of $\Delta ABC=(1+\sqrt{15})$ cm^2

Area of $\Delta ABC=\frac{1}{2}\times$ base $\times$ height

$(1+\sqrt{15})=\frac{1}{2}\times(\sqrt{3}+\sqrt{5})\times\overline{AB}$

$(2+2\sqrt{15})=(\sqrt{3}+\sqrt{5})\times\overline{AB}$

$\overline{AB}=\frac{2+2\sqrt{15}}{\sqrt{3}+\sqrt{5}}=\frac{2+2\sqrt{15}}{\sqrt{3}+\sqrt{5}}\times\frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}-\sqrt{5}}=\frac{2\sqrt{3}-2\sqrt{5}+2\sqrt{45}-2\sqrt{75}}{(\sqrt{3})^2-(\sqrt{5})^2}$

$\overline{AB}=\frac{2\sqrt{3}-2\sqrt{5}+6\sqrt{5}-10\sqrt{3}}{3-5}=\frac{-8\sqrt{3}+4\sqrt{5}}{-2}=\mathbf{(4\sqrt{3}-2\sqrt{5})}$

3. A rectangle has sides of length $2+\sqrt{18}$ m and $\left(5-\frac{4}{\sqrt{2}}\right)$m. Express the area of the rectangle in the form $a+b\sqrt{2}$, where a and b are integers.

Solution

Area $=L\times W=(2+\sqrt{18})\times\left(5-\frac{4}{\sqrt{2}}\right)=10-\frac{8}{\sqrt{2}}+5\sqrt{18}-\sqrt{18}\left(\frac{4}{\sqrt{2}}\right)$

Area $=10-\frac{4\times2}{\sqrt{2}}+5\sqrt{9\times2}-4\sqrt{\frac{18}{2}}=10-4\sqrt{2}+5\times3\sqrt{2}-4\sqrt{9}$

Area $=10-4\sqrt{2}+15\sqrt{2}-12=\mathbf{(11\sqrt{2}-2)}$ m^2

4. Find two numbers whose sum is 68 and difference is 22.

Solution

Let x equal the first number and y equal the second number. Then

According to condition: $x + y = 68$ and $x - y = 22$

$x + y = 68$ $x - y = 22$ adding both $\boldsymbol{x = 45}$	$x + y = 68$ $-x \mp y = 22$ subtracting both $\boldsymbol{y = 23}$

5. The weather in Lahore was unusually warm during the summer of 2024. The TV news reported temperature as high as $48°C$. By using the formula, $(°F = \frac{9}{5}°C + 32)$ find the temperature as Fahrenheit scale.

Solution

°F = 9/5**°C** + 32

°F = 9/5× 48**°C** + 32 = **118.4°F**

6. The sum of the ages of the father and son is 72 years. Six years ago, the father's age was 2 times the age of the son. What was son's age six years ago?

Solution

Son's current age = x year

Father's current age = 72 − x year

Six years ago, Son's age = x − 6 year

Six years ago, Father's age = (72 – x) – 6 = 66 – x year

Six years ago, according to condition: $66 - x = 2(x - 6)$

Simplifying we get: $x = 26$

Six years ago, Son's age = 26 − 6 = **20 year**

7. Mirha sells a toy for Rs. 1520. What will the selling price be to get a 15% profit?

Solution

$\text{CP} = \text{Rs.}\,1520$

$\text{Profit\%} = 15\%$

$\text{Profit} = 15\%\text{ of }1520 = \frac{15}{100} \times 1520 = \text{Rs.}\,228$

$\text{SP} = \text{CP} + \text{Profit}$

$\text{SP} = \text{Rs.}\,1520 + \text{Rs.}\,228$

$\mathbf{SP = Rs.\,1748}$

8. The annual income of Tayyab is Rs. 9,60,000, while the exempted amount is Rs. 1,30,000. How much tax would he have to pay at the rate of 0.75%?

Solution

Taxable Income = Total Income $-$ Exempted Amount

Taxable Income = $\text{Rs.}\,960000 - \text{Rs.}\,130000$

Taxable Income = $\text{Rs.}\,830000$

Tax Rate = $0.75\% = 0.0075$

Tax Amount = Taxable Income $\times$ Tax Rate

Tax Amount = $\text{Rs.}\,830000 \times 0.0075$

Tax Amount = $\mathbf{Rs.\,6225}$

9. Find the compound markup on Rs. 3,75,000 for one year at the rate of 14% compounded annually.

Solution

Principal Amount (P) = $\text{Rs.}\,375000$

Rate of Interest (R) = $14\% = 0.14$

Time (T) = 1 year

Compound Interest (CI) = $\text{P} \times (1 + \text{R})^{\text{T}} - \text{P}$

Compound Interest (CI) = $\text{Rs.}\,375000 \times (1 + 0.14)^{1} - \text{Rs.}\,375000$

Compound Interest (CI) = $\mathbf{Rs.\,52500}$

REVIEW EXERCISE 1

1. Four options are given against each statement. Encircle the correct option.

(i) $\sqrt{7}$ is:

(a) integer (b) rational number

(c)✓ irrational number (d) natural number

(ii) π and e are:

(a) natural numbers (b) integers

(c) rational numbers (d)✓ irrational numbers

(iii) If n is not a perfect square, then $\sqrt{n}$ is:

(a) rational number (b) natural number

(c) integer (d)✓ irrational number

(iv) $\sqrt{3}+\sqrt{5}$ is:

(a) whole number (b) integer

(c) rational number (d)✓ irrational number

(v) For all $x \in R$, $x = x$ is called:

(a)✓ reflexive property (b) transitive number

(c) symmetric property (d) trichotomy property

(vi) Let $a, b, c \in R$, then $a > b$ and $b > c \Rightarrow a > c$ is called __________ property.

(a) trichotomy (b)✓ transitive

(c) additive (d) multiplicative

(vii) $2^x \times 8^x = 64$ then $x =$

(a)✓ $\frac{3}{2}$ (b) $\frac{3}{4}$ (c) $\frac{5}{6}$ (d) $\frac{2}{3}$

(viii) Let $a, b \in R$, then $a = b$ and $b = a$ is called __________ property.

(a) reflexive (b)✓ symmetric

(c) transitive (d) additive

(ix) $\sqrt{75}+\sqrt{27}=$

(a) $\sqrt{102}$ (b) $9\sqrt{3}$ (c) $5\sqrt{3}$ (d)✓ $8\sqrt{3}$

(x) The product of $(3+\sqrt{5})(3-\sqrt{5})$ is:

(a) prime number (b) odd number

(c) irrational number (d)✓ rational number

2. If $a=\frac{3}{2}$, $b=\frac{5}{3}$ and $c=\frac{7}{5}$, then verify that

(i) $a(b+c)=ab+ac$ (ii) $(a+b)c=ac+bc$

Solution

i. $\boldsymbol{a(b+c)=ab+ac}$

$$\text{L.H.S}=\text{a(b + c)}=\frac{3}{2}\left(\frac{5}{3}+\frac{7}{5}\right)=\frac{3}{2}\left(\frac{25+21}{15}\right)=\frac{3}{2}\left(\frac{46}{15}\right)=\frac{138}{30}=\frac{23}{5}$$

$$\text{R.H.S}=ab+ac=\frac{3}{2}\left(\frac{5}{3}\right)+\frac{3}{2}\left(\frac{7}{5}\right)=\frac{15}{6}+\frac{21}{10}=\frac{5}{2}+\frac{21}{10}=\frac{46}{10}=\frac{23}{5}$$

Hence $a(b+c)=ab+ac$

ii. $\boldsymbol{(a+b)c=ac+bc}$

$$\text{L.H.S}=(a+b)c=\left(\frac{3}{2}+\frac{5}{3}\right)\frac{7}{5}=\left(\frac{9+10}{6}\right)\frac{7}{5}=\left(\frac{19}{6}\right)\frac{7}{5}=\frac{133}{30}$$

$$\text{R.H.S}=ac+bc=\left(\frac{3}{2}\right)\frac{7}{5}+\left(\frac{5}{3}\right)\frac{7}{5}=\frac{21}{10}+\frac{35}{15}=\frac{21}{10}+\frac{7}{3}=\frac{133}{30}$$

Hence $(a+b)c=ac+bc$

3. If $a=\frac{4}{3}$, $b=\frac{5}{2}$, $c=\frac{7}{4}$, then verify the associative property of real numbers w.r.t addition and multiplication.

Solution

We have to verify

$(a+b)+c=a+(b+c)$ and $(a\times b)\times c=a\times(b\times c)$

i. $\boldsymbol{(a+b)+c=a+(b+c)}$

$$\text{L.H.S}=(a+b)+c=\left(\frac{4}{3}+\frac{5}{2}\right)+\frac{7}{4}=\left(\frac{8+15}{6}\right)+\frac{7}{4}=\frac{23}{6}+\frac{7}{4}=\frac{67}{12}$$

$$\text{R.H.S}=a+(b+c)=\frac{4}{3}+\left(\frac{5}{2}+\frac{7}{4}\right)=\frac{4}{3}+\left(\frac{10+7}{4}\right)=\frac{4}{3}+\frac{17}{4}=\frac{67}{12}$$

Hence $(a+b)+c=a+(b+c)$

ii. $\boldsymbol{(a\times b)\times c=a\times(b\times c)}$

$$\text{L.H.S}=(a\times b)\times c=\left(\frac{4}{3}\times\frac{5}{2}\right)\times\frac{7}{4}=\frac{20}{6}\times\frac{7}{4}=\frac{10}{3}\times\frac{7}{4}=\frac{70}{12}=\frac{35}{6}$$

$$\text{R.H.S}=a\times(b\times c)=\frac{4}{3}\times\left(\frac{5}{2}\times\frac{7}{4}\right)=\frac{4}{3}\times\frac{35}{8}=\frac{140}{24}=\frac{35}{6}$$

Hence $(a\times b)\times c=a\times(b\times c)$

4. Is 0 a rational number? Explain.

Solution

Yes, zero is a rational number. A rational number is defined as a number that can be expressed as the ratio of two integers, i.e., $\frac{a}{b}$, where a and b are integers and b is non-zero. Zero can be expressed as a ratio of two integers, such as: 0 = 0/1

In this case, both 0 and 1 are integers, and 1 is non-zero. Therefore, zero meets the definition of a rational number.

5. State trichotomy property of real numbers.

Solution

For any two real numbers a and b, exactly one of the following is true:

1. $a < b$ 2. $a = b$ 3. $a > b$

6. Find two rational numbers between 4 and 5.

Solution

$q_1 = \frac{1}{2}(4+5) = \frac{9}{2}$ and $q_2 = \frac{1}{2}\left(\frac{9}{2}+5\right) = \frac{1}{2}\left(\frac{19}{2}\right) = \frac{19}{4}$

Hence required rational are $\mathbf{\frac{9}{2}, \frac{19}{4}}$

7. Simplify the following:

(i) $\sqrt[5]{\frac{x^{15}y^{35}}{z^{20}}}$ (ii) $\sqrt[3]{(27)^{2x}}$ (iii) $\frac{6(3)^{n+2}}{3^{n+1}-3^n}$

Solution

i. $\sqrt[5]{\frac{x^{15}y^{35}}{z^{20}}} = \left(\frac{x^{15}y^{35}}{z^{20}}\right)^{\frac{1}{5}} = \frac{x^{15\times\frac{1}{5}}y^{35\times\frac{1}{5}}}{z^{20\times\frac{1}{5}}} = \boldsymbol{\frac{x^3y^7}{z^4}}$

ii. $\sqrt[3]{(27)^{2x}} = (27)^{\frac{2x}{3}} = (3^3)^{\frac{2x}{3}} = \boldsymbol{3^{2x}}$

iii. $\frac{6(3)^{n+2}}{(3)^{n+1}-3^n} = \frac{3^n(6\times3^2)}{3^n(3-1)} = \frac{6\times9}{2} = \mathbf{27}$

8. The sum of three consecutive odd integers is 51. Find the three integers.

Solution

Let the three consecutive odd integers be x, x+2, and x+4.

$x + (x+2) + (x+4) = 51$

$3x + 6 = 51$

$3x = 45$

$x = 15$

Now that we know x, we can find the other two integers:

$x+2 = 17$

$x+4 = 19$

So, the three consecutive integers are **15**, **17**, and **19**.

9. Abdullah picked up 96 balls and placed them into two buckets. One bucket has twenty-eight more balls than the other bucket. How many balls were in each bucket?

Solution

Let's say the number of balls in the smaller bucket is x. Since the other bucket has 28 more balls, the number of balls in the larger bucket is x + 28.

We know that the total number of balls is 96, so we can set up the equation:

$x + (x + 28) = 96$

$2x + 28 = 96$

$2x = 68$

$x = 34$

So, the smaller bucket has 34 balls.

The larger bucket has 34 + 28 = 62 balls.

Therefore, the two buckets have **34** and **62** balls, respectively.

10. Salma invested Rs. 3,50,000 in a bank, which paid simple profit at the rate of $7\frac{1}{4}\%$ per annum. After 2 years, the rate was increased to 8% per annum. Find the amount she had at the end of 7 years.

Solution

Initial Investment = Rs. 3,50,000

Rate of interest for the first 2 years = $7\frac{1}{4}\%$ = 7.25% per annum

Interest for the first 2 years = (3,50,000 × 7.25% × 2) = Rs. 50,750

Rate of interest for the next 5 years = 8% per annum

Interest for the next 5 years = (3,50,000 × 8% × 5) = Rs. 1,40,000

Amount after 7 years = 3,50,000 + 50,750+ 1,40,000 = Rs. 5,40,750

Therefore, Salma had **Rs. 5,40,750** at the end of 7 years.

Unit 2 Logarithms

EXERCISE 2.1

1. Express the following numbers in scientific notation:

(i)	2000000	(ii)	48900	(iii)	0.0042
(iv)	0.0000009	(v)	73×10^3	(vi)	0.65×10^2

Solution

(i) 2×10^6 (ii) 4.89×10^4 (iii) 4.2×10^{-3} (iv) 9×10^{-7} (v) 7.3×10^4

(vi) 6.5×10^1

2. Express the following numbers in ordinary notation:

(i)	8.04×10^2	(ii)	3×10^5	(iii)	1.5×10^{-2}
(iv)	1.77×10^7	(v)	5.5×10^{-6}	(vi)	4×10^{-5}

Solution

(i) 804 (ii) 300000 (iii) 0.015 (iv) 17700000 (v) 0.0000055 (vi) 0.00004

3. The speed of light is approximately 3×10^8 metres per second. Express it in standard form.
4. The circumference of the Earth at the equator is about 40075000 metres. Express this number in scientific notation.
5. The diameter of Mars is 6.7779×10^3 km. Express this number in standard form.
6. The diameter of Earth is about 1.2756×10^4 km. Express this number in standard form.

Solution

3. 300,000,000 m/sec **4.** 4.0075×10^7 m

5. 6779 km **6.** 12756 km

EXERCISE 2.2

1. Express each of the following in logarithmic form:

(i)	$10^3 = 1000$	(ii)	$2^8 = 256$	(iii)	$3^{-3} = \frac{1}{27}$
(iv)	$20^2 = 400$	(v)	$16^{-\frac{1}{4}} = \frac{1}{2}$	(vi)	$11^2 = 121$
(vii)	$p = q^r$	(viii)	$(32)^{\frac{-1}{5}} = \frac{1}{2}$		

Solution: $\mathbf{\log_b(x) = y \Leftrightarrow b^y = x \quad ; b > 0, x > 0, b \neq 1}$

(i) $\log_{10} 1000 = 3$ (ii) $\log_2 256 = 8$ (iii) $\log_3 \frac{1}{27} = -3$ (iv) $\log_{20} 400 = 2$

(v) $\log_{16} \frac{1}{2} = -\frac{1}{4}$ (vi) $\log_{11} 121 = 2$ (vii) $\log_q p = r$ (viii) $\log_{32} \frac{1}{2} = -\frac{1}{5}$

2. Express each of the following in exponential form:

(i)	$\log_5 125 = 3$	(ii)	$\log_2 16 = 4$	(iii)	$\log_{23} 1 = 0$
(iv)	$\log_5 5 = 1$	(v)	$\log_2 \frac{1}{8} = -3$	(vi)	$\frac{1}{2} = \log_9 3$
(vii)	$5 = \log_{10} 100000$	(viii)	$\log_4 \frac{1}{16} = -2$		

Solution: $\mathbf{\log_b(x) = y \Leftrightarrow b^y = x \quad ; b > 0, x > 0, b \neq 1}$

(i) $5^3 = 125$ (ii) $2^4 = 16$ (iii) $23^0 = 1$ (iv) $5^1 = 5$

(v) $2^{-3} = \frac{1}{8}$ (vi) $9^{\frac{1}{2}} = 3$ (vii) $10^5 = 100000$ (viii) $4^{-2} = \frac{1}{16}$

3. Find the value of x in each of the following:

(i)	$\log_x 64 = 3$	(ii)	$\log_5 1 = x$	(iii)	$\log_x 8 = 1$
(iv)	$\log_{10} x = -3$	(v)	$\log_4 x = \frac{3}{2}$	(vi)	$\log_2 1024 = x$

Solution: $\mathbf{\log_b(x) = y \Leftrightarrow b^y = x \quad ; b > 0, x > 0, b \neq 1}$

i. $\log_x 64 = 3 \Rightarrow x^3 = 64 \Rightarrow x^3 = 4^3 \Rightarrow \mathbf{x = 4}$

ii. $\log_5 1 = x \Rightarrow 5^x = 1 \Rightarrow 5^x = 5^0 \Rightarrow \mathbf{x = 0}$

iii. $\log_x 8 = 1 \Rightarrow x^1 = 8 \Rightarrow \mathbf{x = 8}$

iv. $\log_{10} x = -3 \Rightarrow 10^{-3} = x \Rightarrow x = \frac{1}{10^3} \Rightarrow \mathbf{x = \frac{1}{1000}}$

v. $\log_4 x = \frac{3}{2} \Rightarrow 4^{\frac{3}{2}} = x \Rightarrow x = (2^2)^{\frac{3}{2}} \Rightarrow x = 2^3 \Rightarrow \mathbf{x = 8}$

vi. $\log_2 1024 = x \Rightarrow 2^x = 1024 \Rightarrow 2^x = 2^{10} \Rightarrow \mathbf{x = 10}$

EXERCISE 2.3

1. Find characteristic of the following numbers:

(i)	5287	(ii)	59.28	(iii)	0.0567
(iv)	234.7	(v)	0.000049	(vi)	145000

Solution

(i) 3 (ii) 1 (iii) −2 (iv) 2 (v) −5 (vi) 5

2. Find logarithm of the following numbers:

(i)	43	(ii)	579	(iii)	1.982
(iv)	0.0876	(v)	0.047	(vi)	0.000354

Solution

i. $\log 43 = 1.6335$ Characteristic $= 1$, Mantissa $= 0.6335$

ii. $\log 579 = 2.7627$ Characteristic $= 2$, Mantissa $= 0.7627$

iii. $\log 19.82 = 1.2971$ Characteristic $= 1$, Mantissa $= 0.2971$

iv. $\log 0.0876 = -2 + 0.9425 = -1.0575$ Characteristic $= -2$, Mantissa $= 0.9425$

v. $\log 0.047 = -2 + 0.6721 = -1.3279$ Characteristic $= -2$, Mantissa $= 0.6721$

vi. $\log 0.000354 = -4 + 0.5490 = -3.4518$ Characteristic $= -4$, Mantissa $= 0.5490$

3. If $\log 3.177 = 0.5019$, then find:

(i)	log 3177	(ii)	log 31.77	(iii)	log 0.03177

Solution

i. $\log 3177 = 3.5019$ Characteristic $= 3$, Mantissa $= 0.5019$

ii. $\log 31.77 = 1.5019$ Characteristic $= 1$, Mantissa $= 0.5019$

iii. $\log 0.03177 = -2 + 0.5019 = -1.4981$ Characteristic $= -2$, Mantissa $= 0.5019$

4. Find the value of x.

(i) $\log x = 0.0065$ (ii) $\log x = 1.192$ (iii) $\log x = -3.434$

(iv) $\log x = -1.5726$ (v) $\log x = 4.3561$ (vi) $\log x = -2.0184$

Solution

i. $\log x = 0.0065 \Rightarrow x = \text{antilog}(0.0065) \Rightarrow x = 1.015$

ii. $\log x = 1.192 \Rightarrow x = \text{antilog}(1.192) \Rightarrow x = 15.56$

iii. $\log x = -3.434 \Rightarrow \log x = -4 + 4 - 3.434 \Rightarrow x = \text{antilog}(\bar{4}.566)$

$\Rightarrow x = 0.0003681$

iv. $\log x = -1.5726 \Rightarrow \log x = -2 + 2 - 1.5726 \Rightarrow x = \text{antilog}(\bar{2}.4274)$

$\Rightarrow x = 0.02675$

v. $\log x = 4.3561 \Rightarrow x = \text{antilog}(4.3561) \Rightarrow x = 2270$

vi. $\log x = -2.0184 \Rightarrow \log x = -3 + 3 - 2.0184 \Rightarrow x = \text{antilog}(\bar{3}.9816)$

$\Rightarrow x = 0.009585$

EXERCISE 2.4

1. Without using calculator, evaluate the following:

(i) $\log_2 18 - \log_2 9$ (ii) $\log_2 64 + \log_2 2$ (iii) $\frac{1}{3}\log_3 8 - \log_3 18$

(iv) $2\log 2 + \log 25$ (v) $\frac{1}{3}\log_4 64 + 2\log_5 25$ (vi) $\log_3 12 + \log_3 0.25$

Solution

i. $\log_2 18 - \log_2 9 = \log_2(2 \times 9) - \log_2 9 = \log_2 2 + \log_2 9 - \log_2 9$

$= \log_2 2 = \mathbf{1}$

ii. $\log_2 64 + \log_2 2 = \log_2(2 \times 2 \times 2 \times 2 \times 2 \times 2) + \log_2 2$

$= \log_2(2^6) + \log_2 2 = 6\log_2 2 + \log_2 2 = 7\log_2 2 = 7(1) = \mathbf{7}$

iii. $\frac{1}{3}\log_3 8 - \log_3 18 = \frac{1}{3}\log_3(2 \times 2 \times 2) - \log_3(2 \times 3 \times 3)$

$= \frac{1}{3}\log_3(2^3) - \log_3(2 \times 3^2) = \frac{3}{3}\log_3 2 - \log_3 2 - 2\log_3 3$

$= \log_3 2 - \log_3 2 - 2\log_3 3 = -2(1) = \mathbf{-2}$

iv. $2\log 2 + \log 25 = 2\log 2 + \log(5^2) = 2\log 2 + 2\log 5 = 2(\log 2 + \log 5)$

$= 2\log(2 \times 5) = 2\log 10 = 2(1) = \mathbf{2}$

v. $\frac{1}{3}\log_4 64 + 2\log_5 25 = \frac{1}{3}\log_4(4^3) + 2\log_5(5^2) = \frac{3}{3}\log_4 4 + 2 \times 2\log_5 5$

$= \log_4 4 + 4\log_5 5 = (1) + 4(1) = 1 + 4 = \mathbf{5}$

vi. $\log_3 12 + \log_3 0.25 = \log_3 12 + \log_3 \frac{25}{100} = \log_3 12 + \log_3 \frac{1}{4} = \log_3 \frac{12}{4}$

$= \log_3 3 = \mathbf{1}$

2. Write the following as a single logarithm:

(i) $\frac{1}{2}\log 25 + 2\log 3$ (ii) $\log 9 - \log\frac{1}{3}$

(iii) $\log_5 b^2.\log_a 5^3$ (iv) $2\log_3 x + \log_3 y$

(v) $4\log_5 x - \log_5 y + \log_5 z$ (vi) $2\ln a + 3\ln b - 4\ln c$

Solution

i. $\frac{1}{2}\log 25 + 2\log 3 = \frac{1}{2}\log(5^2) + \log(3^2) = \log 5 + \log 9 = \log(5\times 9) = \mathbf{\log 45}$

ii. $\log 9 - \log\frac{1}{3} = \log\left(\frac{9}{\frac{1}{3}}\right) = \log(9\times 3) = \mathbf{\log 27}$

iii. $\log_5 b^2.\log_a 5^3 = 2\log_5 b \times 3\log_a 5 = 2\frac{\log_a b}{\cancel{\log_a}5}\times 3\frac{\log_a 5}{\cancel{\log_a}a} = 6\frac{\log_a b}{(1)} = \mathbf{6\log_a b}$

vi. $2\log_3 x + \log_3 y = \log_3(x^2) + \log_3 y = \mathbf{\log_3 x^2 y}$

vi. $2\ln a + 3\ln b - 4\ln c = \ln a^2 + \ln b^3 - \ln c^4 = \mathbf{\ln\frac{a^2b^3}{c^4}}$

3. Expand the following using laws of logarithms:

(i) $\log\left(\frac{11}{5}\right)$ (ii) $\log_5\sqrt{8a^6}$ (iii) $\ln\left(\frac{a^2b}{c}\right)$

(iv) $\log\left(\frac{xy}{z}\right)^{\frac{1}{9}}$ (v) $\ln\sqrt[3]{16x^3}$ (vi) $\log_2\left(\frac{1-a}{b}\right)^5$

Solution

i. $\log\left(\frac{11}{5}\right) = \mathbf{\log 11 - \log 5}$

ii. $\log_5\sqrt{8a^6} = \log_5(2^3\times a^6)^{\frac{1}{2}} = \log_5\left(2^{\frac{3}{2}}\times a^3\right) = \mathbf{\frac{3}{2}\log_5 2 + 3\log_5 a}$

iii. $\ln\left(\frac{a^2b}{c}\right) = \ln a^2 + \ln b - \ln c = \mathbf{2\ln a + \ln b - \ln c}$

iv. $\ln\left(\frac{xy}{z}\right)^{\frac{1}{9}} = \mathbf{\frac{1}{9}}\ln\left(\frac{xy}{z}\right) = \mathbf{\frac{1}{9}[\ln x + \ln y - \ln z]}$

v. $\ln\sqrt[3]{16x^3} = \ln(2^4\times x^3)^{\frac{1}{3}} = \ln\left(2^{\frac{4}{3}}\times x\right) = \mathbf{\frac{4}{3}\ln 2 + \ln x}$

vi. $\log_2\left(\frac{1-a}{b}\right)^5 = \mathbf{5}\log_2\left(\frac{1-a}{b}\right) = \mathbf{5[\log_2(1-a) - \log_2 b]}$

4. Find the value of x in the following equations:

(i) $\log 2+\log x=1$ (ii) $\log_2 x+\log_2 8=5$

(iii) $(81)^x=(243)^{x+2}$ (iv) $\left(\frac{1}{27}\right)^{x-6}=27$

(v) $\log(5x-10)=2$ (vi) $\log_2(x+1)-\log_2(x-4)=2$

Solution

i. $\log 2+\log x=1 \Rightarrow \log 2x=\log 10 \Rightarrow 2x=10 \Rightarrow \mathbf{x=5}$

ii. $\log_2 x+\log_2 8=5 \Rightarrow \log_2 x+\log_2 8=5\log_2 2 \Rightarrow \log_2 8x=\log_2 2^5 \Rightarrow 8x=32 \Rightarrow \mathbf{x=4}$

iii. $(81)^x=(243)^{x+2} \Rightarrow (3^4)^x=(3^5)^{x+2} \Rightarrow 3^{4x}=3^{5x+10} \Rightarrow 5x+10=4x \Rightarrow \mathbf{x=-10}$

iv. $\left(\frac{1}{27}\right)^{x-6}=27 \Rightarrow (3^{-3})^{x-6}=3^3 \Rightarrow 3^{-3x+18}=3^3 \Rightarrow -3x+18=3 \Rightarrow \mathbf{x=5}$

v. $\log(5x-10)=2 \Rightarrow \log(5x-10)=2\log 10 \Rightarrow \log(5x-10)=\log 10^2$

$\Rightarrow 5x-10=100 \Rightarrow 5x=110 \Rightarrow \mathbf{x=22}$

vi. $\log_2(x+1)-\log_2(x-4)=2 \Rightarrow \log_2\left(\frac{x+1}{x-4}\right)=2\log_2 2$

$\Rightarrow \log_2\left(\frac{x+1}{x-4}\right)=\log_2 2^2 \Rightarrow \frac{x+1}{x-4}=4 \Rightarrow x+1=4x-16$

$\Rightarrow 3x=17 \Rightarrow x=\frac{17}{3} \Rightarrow \mathbf{x=5\frac{2}{3}}$

5. Find the values of the following with the help of logarithm table:

(i) $\dfrac{3.68\times 4.21}{5.234}$ (ii) $4.67\times 2.11\times 2.397$

(iii) $\dfrac{(20.46)^2\times(2.4122)}{754.3}$ (iv) $\dfrac{\sqrt[3]{9.364}\times 21.64}{3.21}$

5(i). $\mathbf{\log\left(\frac{3.68\times 4.21}{5.234}\right)}$ $=???$

Solution

Let $x=\frac{3.68\times 4.21}{5.234}$

$\log x=\log\left(\frac{3.68\times 4.21}{5.234}\right)$ taking logarithm on both sides

$\log x=\log(3.68)+\log(4.21)-\log(5.234)$

$\log x=0.5658+0.6243-0.7188$

$\log x=0.4713$

$x=\text{antilog}(0.4713)$

$\Rightarrow \mathbf{\log\left(\frac{3.68\times 4.21}{5.234}\right)=2.960}$

5(ii). $\mathbf{log(4.67 \times 2.11 \times 2.397)} = ???$

Solution

Let $x = 4.67 \times 2.11 \times 2.397$

$\log x = \log(4.67 \times 2.11 \times 2.397)$ taking logarithm on both sides

$\log x = \log(4.67) + \log(2.11) + \log(2.397)$

$\log x = 0.6693 + 0.3243 + 0.3797$

$\log x = 1.3733$

$x = \text{antilog}(1.3733)$

$\Rightarrow \mathbf{log(4.67 \times 2.11 \times 2.397) = 23.62}$

5(iii). $\mathbf{log\left[\frac{(20.46)^2 \times (2.4122)}{754.3}\right]} = ???$

Solution

Let $x = \frac{(20.46)^2 \times (2.4122)}{754.3}$

$\log x = \log\left[\frac{(20.46)^2 \times (2.4122)}{754.3}\right]$ taking logarithm on both sides

$\log x = 2\log(20.46) + \log(2.4122) - \log(754.3)$

$\log x = 2(1.3109) + 0.3824 - 2.8776$

$\log x = 0.1266$

$x = \text{antilog}(0.1266)$

$\Rightarrow \mathbf{log\left[\frac{(20.46)^2 \times (2.4122)}{754.3}\right] = 1.339}$

5(iv). $\mathbf{log\left[\frac{\sqrt[3]{9.364} \times (21.64)}{3.21}\right]} = ???$

Solution

Let $x = \frac{\sqrt[3]{9.364} \times (21.64)}{3.21}$

$\log x = \log\left[\frac{\sqrt[3]{9.364} \times (21.64)}{3.21}\right]$ taking logarithm on both sides

$\log x = \frac{1}{3}\log(9.364) + \log(21.64) - \log(3.21)$

$\log x = \frac{1}{3}(0.9715) + 1.3353 - 0.5065$

$\log x = 1.1526$

$x = \text{antilog}(1.1526)$

$\Rightarrow \mathbf{log\left[\frac{\sqrt[3]{9.364} \times (21.64)}{3.21}\right] = 14.21}$

6. The formula to measure the magnitude of earthquakes is given by $M = \log_{10}\left(\frac{A}{A_o}\right)$. If amplitude ($A$) is 10,000 and reference amplitude (A_o) is 10. What is the magnitude of the earthquake?

Solution

$$\mathbf{M = \log_{10}\left[\frac{A}{A_0}\right] = \log_{10}\left[\frac{10000}{10}\right]} = ???$$

$$M = \log_{10}\left[\frac{10000}{10}\right] \Rightarrow M = \log_{10}[1000] \Rightarrow M = \log_{10}[10^3] \Rightarrow \mathrm{M} = 3\log_{10}(10)$$

$$\Rightarrow \mathbf{M = \log_{10}\left[\frac{A}{A_0}\right] = \log_{10}\left[\frac{10000}{10}\right] = 3\ \ rector\ scale}$$

7. Abdullah invested Rs. 100,000 in a saving scheme and gains interest at the rate of 5% per annum so that the total value of this investment after t years is Rs y. This is modelled by an equation $y = 100{,}000\,(1.05)^t$, $t \geq 0$. Find after how many years the investment will be double.

Solution

Initial investment = Rs. 100000

Interest rate = 5% per annum

Total value after t years = y

The equation modeling this situation is:

$$\mathrm{y} = 100000 \times (1.05)^{\mathrm{t}}$$

We want to find years when the investment will be double, i.e., y = 2,00,000

$$2{,}00{,}000 = 1{,}00{,}000 \times (1.05)^{\mathrm{t}}$$

$$2 = (1.05)^t \Rightarrow \log 2 = \log(1.05)^{\mathrm{t}} \Rightarrow \log 2 = \mathrm{t} \times \log(1.05)$$

$$\Rightarrow \mathrm{t} = \frac{\log 2}{\log(1.05)} \Rightarrow t = \frac{0.3010}{0.0212} \Rightarrow \mathbf{t \approx 14.21\ years}$$

8. Huria is hiking up a mountain where the temperature (T) decreases by 3% (or a factor of 0.97) for every 100 metres gained in altitude. The initial temperature (T_i) at sea level is 20°C. Using the formula $T = T_i \times 0.97^{\frac{h}{100}}$, calculate the temperature at an altitude (h) of 500 metres.

Solution

First method	Second method
$\mathrm{T} = \mathrm{T_i} \times (0.97)^{\frac{h}{100}}$	$\mathrm{T} = \mathrm{T_i} \times (0.97)^{\frac{h}{100}}$
$\mathrm{T} = 20 \times (0.97)^{\frac{500}{100}}$	$\mathrm{T} = 20 \times (0.97)^{\frac{500}{100}}$
$\mathrm{T} = 20 \times (0.97)^5$	$\log \mathrm{T} = \log(20 \times (0.97)^5)$
$\mathrm{T} = 20 \times 0.859$	$\log \mathrm{T} = \log 20 + 5\log(0.97) = 1.3011 + 5(-0.0133)$
$\mathbf{T \approx 17.17°C}$	$\log \mathrm{T} = 1.2346 \Rightarrow \mathrm{T} = \mathrm{antilog}(1.2346)$
	$\mathbf{T \approx 17.17°C}$

REVIEW EXERCISE 2

1. Four options are given against each statement. Encircle the correct option.

(i) The standard form of 5.2×10^6 is:

(a) 52,000 (b) 520,000 (c) ✓ 5,200,000 (d) 52,000,000

(ii) Scientific notation of 0.00034 is:

(a) 3.4×10^3 (b) ✓ 3.4×10^{-4} (c) 3.4×10^4 (d) 3.4×10^{-3}

(iii) The base of common logarithm is:

(a) 2 (b) ✓ 10 (c) 5 (d) e

(iv) $\log_2 2^3 =$ __________.

(a) 1 (b) 2 (c) 5 (d) ✓ 3

(v) $\log 100 =$ __________.

(a) ✓ 2 (b) 3 (c) 10 (d) 1

(vi) If $\log 2 = 0.3010$, then $\log 200$ is:

(a) 1.3010 (b) 0.6010 (c) ✓ 2.3010 (d) 2.6010

(vii) $\log (0) =$ __________.

(a) positive (b) negative (c) zero (d) ✓ undefined

(viii) $\log 10{,}000 =$

(a) 2 (b) 3 (c) ✓ 4 (d) 5

(ix) $\log 5 + \log 3 =$ __________.

(a) $\log 0$ (b) $\log 2$ (c) $\log\left(\frac{5}{3}\right)$ (d) ✓ $\log 15$

(x) $3^4 = 81$ in logarithmic form is:

(a) $\log_3 4 = 81$ (b) $\log_4 3 = 81$

(c) ✓ $\log_3 81 = 4$ (d) $\log_4 81 = 3$

2. Express the following numbers in scientific notation:

(i) 0.000567 (ii) 734 (iii) 0.33×10^3

Solution

(i) 5.67×10^{-4} (ii) 7.34×10^2 (iii) 3.3×10^2

3. Express the following numbers in ordinary notation:

(i) 2.6×10^3 (ii) 8.794×10^{-4} (iii) 6×10^{-6}

Solution

(i) 2600 (ii) 0.0008794 (iii) 0.000006

4. Express each of the following in logarithmic form:

(i) $3^7 = 2187$ (ii) $a^b = c$ (iii) $(12)^2 = 144$

Solution

(i) $\log_3 2187 = 7$ (ii) $\log_a c = b$ (iii) $\log_{12} 144 = 2$

5. Express each of the following in exponential form:

(i) $\log_4 8 = x$ (ii) $\log_9 729 = 3$ (iii) $\log_4 1024 = 5$

Solution

(i) $4^x = 8$ (ii) $9^3 = 729$ (iii) $4^5 = 1024$

6. Find value of x in the following:

(i) $\log_9 x = 0.5$ (ii) $\left(\frac{1}{9}\right)^{3x} = 27$ (iii) $\left(\frac{1}{32}\right)^{2x} = 64$

Solution

i. $\log_9 x = 0.5 \Rightarrow x = 9^{0.5} \Rightarrow x = (3^2)^{\frac{1}{2}} \Rightarrow \mathbf{x = 3}$

ii. $\left(\frac{1}{9}\right)^{3x} = 27 \Rightarrow \left(\frac{1}{3^2}\right)^{3x} = 3^3 \Rightarrow (3^{-2})^{3x} = 3^3 \Rightarrow 3^{-6x} = 3^3$

$\Rightarrow -6x = 3 \Rightarrow x = -\frac{3}{6} \Rightarrow \mathbf{x = -\frac{1}{2}}$

iii. $\left(\frac{1}{32}\right)^{2x} = 64 \Rightarrow \left(\frac{1}{2^5}\right)^{2x} = 2^6 \Rightarrow (2^{-5})^{2x} = 2^6 \Rightarrow 2^{-10x} = 2^6$

$\Rightarrow -10x = 6 \Rightarrow x = -\frac{6}{10} \Rightarrow \mathbf{x = -\frac{3}{5}}$

7. Write the following as a single logarithm:

(i) $7\log x - 3\log y^2$ (ii) $3\log 4 - \log 32$

(iii) $\frac{1}{3}(\log_5 8 + \log_5 27) - \log_5 3$

Solution

i. $7\text{logx} - 3\log y^2 = \text{logx}^7 - \log y^6 = \mathbf{log\frac{x^7}{y^6}}$

ii. $3\log 4 - \log 32 = \log 4^3 - \log 32 = \log\frac{4^3}{32} = \log\frac{64}{32} = \mathbf{log2}$

iii. $\frac{1}{3}(\log_5 8 + \log_5 27) - \log_5 3 = \frac{1}{3}[\log_5(8 \times 27)] - \log_5 3$

$= \frac{1}{3}[\log_5(216)] - \log_5 3 = \log_5(6^3)^{\frac{1}{3}} - \log_5 3$

$= \log_5 6 - \log_5 3 = \log_5\frac{6}{3} = \mathbf{log_5 2}$

8. Expand the following using laws of logarithms:

(i) $\log(x\,y\,z^6)$ (ii) $\log_3 \sqrt[6]{m^5 n^3}$ (iii) $\log\sqrt{8x^3}$

Solution

i. $\log(xyz^6) = \text{logx} + \text{logy} + \text{logz}^6 = \mathbf{logx + logy + 6logz}$

ii. $\log_3 \sqrt[6]{m^5n^3} = \log_3(m^5n^3)^{\frac{1}{6}} = \frac{1}{6}[\log_3 m^5 + \log_3 n^3] = \mathbf{\frac{1}{6}[5log_3 m + 3log_3 n]}$

iii. $\log\sqrt{8x^3} = \log(8x^3)^{\frac{1}{2}} = \log(2^3x^3)^{\frac{1}{2}} = \log(2\text{x})^{\frac{3}{2}} = \mathbf{\frac{3}{2}[log2 + logx]}$

9. Find the values of the following with the help of logarithm table:

(i) $\sqrt[3]{68.24}$ (ii) 319.8×3.543 (iii) $\frac{36.12 \times 750.9}{113.2 \times 9.98}$

9(i). $\mathbf{log[\sqrt[3]{68.24}]}$ =???

Solution

Let $x = \sqrt[3]{68.24} = (68.24)^{\frac{1}{3}}$

$\log x = \log(68.24)^{\frac{1}{3}}$ taking logarithm on both sides

$\log x = \frac{1}{3}\log(68.24) = \frac{1}{3}(1.8340)$

$\log x = 0.6113$

$x = \text{antilog}(0.6113)$

$\Rightarrow \mathbf{log[\sqrt[3]{68.24}] = 4.086}$

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9(ii). $\mathbf{log(319.8 \times 3.543)}$ =???

Solution

Let $x = 319.8 \times 3.543$

$\log x = \log(319.8 \times 3.543)$ taking logarithm on both sides

$\log x = \log(319.8) + \log(3.543)$

$\log x = 2.5049 + 0.5494$

$\log x = 3.0543$

$x = \text{antilog}(3.0543)$

$\Rightarrow \mathbf{\log(319.8 \times 3.543) = 1133}$

9(iii). $\mathbf{\log\left(\frac{36.12\times750.9}{113.2\times9.98}\right)}$ =???

Solution

Let $x = \frac{36.12\times750.9}{113.2\times9.98}$

$\log x = \log\left(\frac{36.12\times750.9}{113.2\times9.98}\right)$ taking logarithm on both sides

$\log x = \log(36.12) + \log(750.9) - \log(113.2) - \log(9.98)$

$\log x = 1.5578 + 2.8756 - 2.0539 - 0.9991$

$\log x = 1.3804$

$x = \text{antilog}(1.3804)$

$\Rightarrow \mathbf{\log\left(\frac{36.12\times750.9}{113.2\times9.98}\right) = 24.01}$

10. In the year 2016, the population of a city was 22 millions and was growing at a rate of 2.5% per year. The function $p(t) = 22(1.025)^t$ gives the population in millions, t years after 2016. Use the model to determine in which year the population will reach 35 millions. Round the answer to the nearest year.

Solution

$P(t) = 22 \times (1.025)^t$

$35 = 22 \times (1.025)^t$ when P(t) = 35

$1.591 = (1.025)^t$ dividing by 22

$\log 1.591 = t \times \log 1.025$ taking logarithm on both sides

$0.2014 = t \times 0.0107 \Rightarrow t = \frac{0.2014}{0.0107}$

$t = 18.81 \approx 19$ years

Since t represents years after 2016, add 19 to 2016:

Year ≈ 2016 + 19 ≈ 2035

Unit 3 Sets and Functions

EXERCISE 3.1

1. Write the following sets in set builder notation:

(i) $\{1, 4, 9, 16, 25, 36, \ldots, 484\}$ (ii) $\{2, 4, 8, 16, 32, 64, \ldots, 150\}$

(iii) $\{0, \pm 1, \pm 2, \ldots, \pm 1000\}$ (iv) $\{6, 12, 18, \ldots, 120\}$

(v) $\{100, 102, 104, \ldots, 400\}$ (vi) $\{1, 3, 9, 27, 81, \ldots\}$

(vii) $\{1, 2, 4, 5, 10, 20, 25, 50, 100\}$ (viii) $\{5, 10, 15, \ldots, 100\}$

(ix) The set of all integers between – 100 and 1000

Solution

(i) $\{x | x = n^2, n \in N \wedge 1 \leq x < 500\}$ **(ii)** $\{x | x = 2^n, n \in N \wedge 2 \leq x \leq 150\}$

(iii) $\{x | x \in Z \wedge 0 \leq x \leq 1000\}$ **(iv)** $\{x | x = 6n, n \in N \wedge 1 \leq n \leq 20\}$

(v) $\{x | x = 100 + 2n, n \in W \wedge 1 \leq n \leq 150\}$ **(vi)** $\{x | x = 3^n, n \in W\}$

(vii) $\{x | x \text{ is a divisor of } 100\}$ **(viii)** $\{x | x = 5n, n \in N \wedge 1 \leq n \leq 20\}$

(ix) $\{x | x \in Z \wedge -100 < x < 1000\}$

2. Write each of the following sets in tabular forms:

(i) $\{x | x \text{ is a multiple of } 3 \wedge x \leq 35\}$ (ii) $\{x | x \in R \wedge 2x + 1 = 0\}$

(iii) $\{x | x \in P \wedge x < 12\}$ (iv) $\{x | x \text{ is a divisor of } 128\}$

(v) $\{x | x = 2^n, n \in N \wedge n < 8\}$ (vi) $\{x | x \in N \wedge x + 4 = 0\}$

(vii) $\{x | x \in N \wedge x = x\}$ (viii) $\{x | x \in Z \wedge 3x + 1 = 0\}$

Solution

(i) $\{3, 6, 9, \ldots, 35\}$ (ii) $\left\{-\frac{1}{2}\right\}$

(iii) $\{2, 3, 5, 7, 11\}$ (iv) $\{1, 2, 4, 8, 16, 32, 64, 128\}$ (v) $\{2,4,8,16,32,64,128\}$

(vi) $\{ \}$ (vii) $\{1, 2, 3, 4, 5, \ldots\}$ (viii) $\{ \}$

3. Write two proper subsets of each of the following sets:

(i) $\{a, b, c\}$ (ii) $\{0, 1\}$ (iii) N (iv) Z

(v) Q (vi) R (vii) $\{x \mid x \in Q \wedge 0 < x \leq 2\}$

Solution

i. The Proper subsets of {a, b, c} are {a}, {b}.
ii. The Proper subsets of {0,1} are { 0 }, { 1 }.
iii. The Proper subsets of N = {1,2,3,… } are {1}, {2}.
iv. The Proper subsets of Z = $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ are {1}, {2}.
v. The Proper subsets of Q are {1}, {2}.
vi. The Proper subsets of R are {1}, {2}.
vii. The Proper subsets of $\{x|x \in Q \wedge 0 < x \leq 2\}$ are {1}, {2}.

4. Is there any set which has no proper subset? If so, name that set.

Solution

Yes, { } or φ

5. What is the difference between $\{a, b\}$ and $\{\{a,b\}\}$?

Solution

{a,b} is a set containing two elements a and b while {{a,b}} is a set containing one element {a,b}.

6. What is the number of elements of the power set of each of the following sets?

(i) { } (ii) {0, 1} (iii) {1, 2, 3, 4, 5, 6, 7}

(iv) {0, 1, 2, 3, 4, 5, 6, 7} (v) $\{a, \{b, c\}\}$

(vi) $\{\{a, b\}, \{b, c\}, \{d, e\}\}$

Solution

(i) 1 (ii) 4 (iii) 128 (iv) 256 (v) 4 (vi) 8

7. Write down the power set of each of the following sets:

(i) {9, 11} (ii) $\{+, -, \times, \div\}$ (iii) $\{\phi\}$ (iv) $\{a, \{b,c\}\}$

Solution

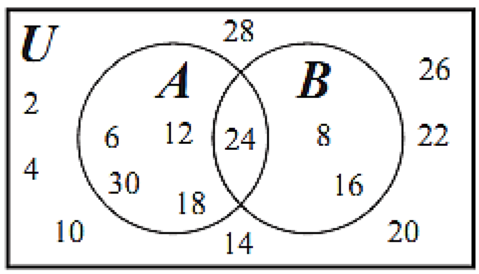
i. The Power set of {9,11} is { φ, {9}, {11},{9,11}}.
ii. The Power set of $\{+, -, \times, \div\}$ is
$\{\varphi, \{+\}, \{-\}, \{\times\}, \{\div\}, \{+,-\}, \{+\ \times\}, \{+,\div\}, \{-,\times\}, \{-,\div\}, \{\times,\div\}, \{+,-,\times\},$
$\{+,-,\div\}, \{+,\times,\div\}, \{-,\times,\div\}, \{+,-,\times,\div\}\}.$
iii. The Power set of {φ} is { φ, {φ}}.
iv. The Power set of $\{a, \{b, c\}\}$ is $\{\varphi, \{a\}. \{\{b, c\}\}, \{a, \{b, c\}\}\}$.

Exercise 3.2

1. Consider the universal set $U = \{x : x \text{ is multiple of } 2 \text{ and } 0 < x \leq 30\}$, $A = \{x : x \text{ is a multiple of } 6\}$ and $B = \{x : x \text{ is a multiple of } 8\}$
   (i) List all elements of sets A and B in tabular form
   (ii) Find $A \cap B$ (iii) Draw a Venn diagram

Solution

(i) $A = \{6, 12, 18, 24, 30\}$, $B = \{8, 16, 24\}$ (ii) $A \cap B = \{24\}$



2. Let, $U = \{x : x \text{ is an integer and } 0 < x \leq 150\}$,
   $G = \{x : x = 2^m \text{ for integer } m \text{ and } 0 \leq m \leq 7\}$ and
   $H = \{x : x \text{ is a square}\}$
   (i) List all elements of sets G and H in tabular form
   (ii) Find $G \cup H$ (iii) Find $G \cap H$

Solution

(i) $G = \{1, 2, 4, 8, 16, 32, 64, 128\}$,
$H = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144\}$
(ii) $G \cup H = \{1, 2, 4, 8, 9, 16, 25, 32, 36, 49, 64, 81, 100, 121, 128, 144\}$
(iii) $G \cap H = \{1, 4, 16, 64\}$

3. Consider the sets $P = \{x : x \text{ is a prime number and } 0 < x \leq 20\}$ and
   $Q = \{x : x \text{ is a divisor of } 210 \text{ and } 0 < x \leq 20\}$
   (i) Find $P \cap Q$ (ii) Find $P \cup Q$

Solution

(i) $P \cap Q = \{2,3,5,7,9,11,13,17,19,20\} \cap \{1, 2, 3, 5, 6, 7, 10, 14, 15\} = \{2,3,5,7\}$
(ii) $P \cup Q = \{2,3,5,7,9,11,13,17,19,20\} \cup \{1, 2, 3, 5, 6, 7, 10, 14, 15\}$
$P \cup Q = \{1, 2, 3, 5, 6, 7, 10,11,13, 14, 15,17,19,20\}$

4. Verify the commutative properties of union and intersection for the following pairs of sets:

(i) $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 6, 8, 10\}$ (ii) N, Z

(iii) $A = \{x \mid x \in R \wedge x \geq 0\}$, $B = R$.

Solution

4.(i) $\boldsymbol{A \cup B = B \cup A}$ **also** $\boldsymbol{A \cap B = B \cap A}$

$A \cup B = \{1,2,3,4,5\} \cup \{4,6,8,10\} = \{1,2,3,4,5,6,8,10\}$

$B \cup A = \{4,6,8,10\} \cup \{1,2,3,4,5\} = \{1,2,3,4,5,6,8,10\}$

Hence $A \cup B = B \cup A$

$A \cap B = \{1,2,3,4,5\} \cap \{4,6,8,10\} = \{4\}$

$B \cap A = \{4,6,8,10\} \cap \{1,2,3,4,5\} = \{4\}$

Hence $A \cap B = B \cap A$

4.(ii) $\boldsymbol{N \cup Z = Z \cup N}$ **also** $\boldsymbol{N \cap Z = Z \cap N}$

$N \cup Z = \{1,2,3, \dots\} \cup \{0, \pm1, \pm2, \pm3, \dots\} = \{0, \pm1, \pm2, \pm3, \dots\}$

$Z \cup N = \{0, \pm1, \pm2, \pm3, \dots\} \cup \{1,2,3, \dots\} = \{0, \pm1, \pm2, \pm3, \dots\}$

Hence $N \cup Z = Z \cup N$

$N \cap Z = \{1,2,3, \dots\} \cap \{0, \pm1, \pm2, \pm3, \dots\} = \{1,2,3, \dots\}$

$Z \cap N = \{0, \pm1, \pm2, \pm3, \dots\} \cap \{1,2,3, \dots\} = \{1,2,3, \dots\}$

Hence $N \cap Z = Z \cap N$

4.(iii) $\boldsymbol{A \cup B = B \cup A}$ **also** $\boldsymbol{A \cap B = B \cap A}$

$A \cup B = \{0,1,2,3,4,5\} \cup R = R$

$B \cup A = R \cup \{0,1,2,3,4,5\} = R$

Hence $A \cup B = B \cup A$

$A \cap B = \{0,1,2,3,4,5\} \cap R = \{0,1,2,3,4,5\}$

$B \cap A = R \cap \{0,1,2,3,4,5\} = \{0,1,2,3,4,5\}$

Hence $A \cap B = B \cap A$

5. Let $U = \{a, b, c, d, e, f, g, h, i, j\}$

$A = \{a, b, c, d, g, h\}$, $B = \{c, d, e, f, j\}$,

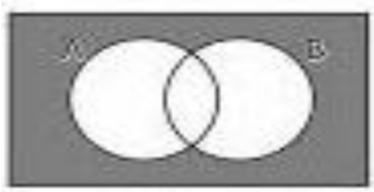
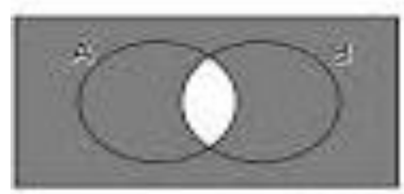
Verify De Morgan's Laws for these sets. Draw Venn diagram

Solution

We have to verify

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

$(\mathbf{A} \cup \mathbf{B})' = \mathbf{A}' \cap \mathbf{B}'$	$(\mathbf{A} \cap \mathbf{B})' = \mathbf{A}' \cup \mathbf{B}'$
A = {a, b, c, d, g, h} B = {c, d, e, f, j} U = {a, b, c, d, e, f, g, h, i, j} A′ = U - A = {a, b, c, d, e, f, g, h, i, j} - {a, b, c, d, g, h} = {e, f, i, j} B′ = U - B = {a, b, c, d, e, f, g, h, i, j} - {c, d, e, f, j} = {a, b, g, h, i} A ∪ B = {a, b, c, d, e, f, g, h, j} (A ∪ B)′ = U - (A ∪ B) = {a, b, c, d, e, f, g, h, i, j} - {a, b, c, d, e, f, g, h, j} = {i} A′ ∩ B′= {e, f, i, j} ∩ {a, b, g, h, i} = {i} 	A = {a, b, c, d, g, h} B = {c, d, e, f, j} U = {a, b, c, d, e, f, g, h, i, j} A′ = U - A = {a, b, c, d, e, f, g, h, i, j} - {a, b, c, d, g, h} = {e, f, i, j} B′ = U - B = {a, b, c, d, e, f, g, h, i, j} - {c, d, e, f, j} = {a, b, g, h, i} A ∩ B = {c, d} (A ∩ B)′ = U - (A ∩ B) = {a, b, c, d, e, f, g, h, i, j} - {c, d} = {a, b, e, f, g, h, i, j} A′ ∪ B′= {e, f, i, j} ∪ {a, b, g, h, i} = {a, b, e, f, g, h, i, j} 

6. If $U = \{1, 2, 3, \ldots, 20\}$ and $A = \{1, 3, 5, \ldots, 19\}$, verify the following:

(i) $A \cup A' = U$ (ii) $A \cap U = A$ (iii) $A \cap A' = \phi$

Solution

$U = \{1,2,3, \ldots ,20\}$ and $A = \{1,3,5, \ldots ,19\}$

$A' = U - A = \{1,2,3, \ldots ,20\} - \{1,3,5, \ldots ,19\} = \{2,4,6, \ldots ,20\}$

(i) $A \cup A' = \{1,3,5, \ldots ,19\} \cup \{2,4,6, \ldots ,20\} = \{1,2,3, \ldots ,20\} = U$

(ii) $A \cap U = \{1,3,5, \ldots ,19\} \cap \{1,2,3, \ldots ,20\} = \{1,3,5, \ldots ,19\} = A$

(iii) $A \cap A' = \{1,3,5, \ldots ,19\} \cap \{2,4,6, \ldots ,20\} = \varphi$

7. In a class of 55 students, 34 like to play cricket and 30 like to play hockey. Also each student likes to play at least one of the two games. How many students like to play both games?

Solution

n (C) = 34 ; n (H) = 30 ; n (U) = 55 ; n (C ∪ H) = 55

n(C ∪ H) = n(C) + n(H) – n(C ∩ H)

$55 = 34 + 30 - n(C \cap H) \Rightarrow 55 = 64 - n(C \cap H)$

$\Rightarrow n(C \cap H) = 64 - 55$

$\Rightarrow \mathbf{n(C \cap H) = 9}.$

8. In a group of 500 employees, 250 can speak Urdu, 150 can speak English, 50 can speak Punjabi, 40 can speak Urdu and English, 30 can speak both English and Punjabi, and 10 can speak Urdu and Punjabi. How many can speak all three languages?

Solution

$n(U \cup E \cup P) = 500$; $n(U) = 250$; $n(E) = 150$; $n(P) = 50$

$n(U \cap E) = 40$; $n(E \cap P) = 30$; $n(U \cap P) = 10$

$n(U \cap E \cap P) = ???$

$n(U \cup E \cup P) = n(U) + n(E) + n(P) - n(U \cap E) - n(E \cap P) - n(U \cap P) + n(U \cap E \cap P)$

$500 = 250 + 150 + 50 - 40 - 30 - 10 + n(U \cap E \cap P)$

$500 = 450 - 80 + n(U \cap E \cap P)$

$500 = 370 + n(U \cap E \cap P)$

$\mathbf{n(U \cap E \cap P) = 130}$

9. In sports events, 19 people wear blue shirts, 15 wear green shirts, 3 wear blue and green shirts, 4 wear a cap and blue shirts, and 2 wear a cap and green shirts. The total number of people with either a blue or green shirt or cap is 25. How many people are wearing caps?

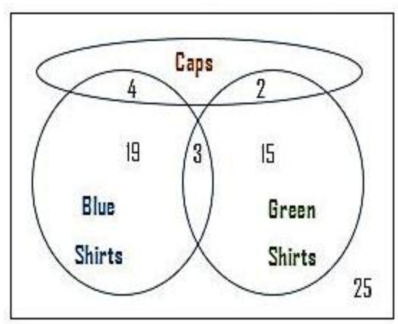
Solution

$n(B) = 19$; $n(G) = 15$; $n(C) = ?$; $n(B \cap G) = 3$; $n(B \cap C) = 4$

$n(G \cap C) = 2$; $n(B \cup G \cup C) = 25$

$n(B \cup G \cup C) = n(B) + n(G) + n(C) - n(B \cap G) - n(B \cap C) - n(G \cap C) + n(B \cap G \cap C)$

$25 = 19 + 15 + n(C) - 3 - 4 - 2 + n(B \cap G \cap C)$



$0 = n(C) + n(B \cap G \cap C)$

As number of element in any set can be zero or positive, which concludes that Sum of number of elements of two sets can only be zero, if both sets are empty.

Hence, $\mathbf{n(C) = 0}$ or number of players wearing only caps are zero.

10. In a training session,17 participants have laptops, 11 have tablets, 9 have laptops and tablets, 6 have laptops and books, and 4 have both tablets and books. Eight participants have all three items. The total number of participants with laptops, tablets, or books is 35. How many participants have books?

Solution

n (L) = 17 ; n (T) = 11 ; n (L ∩ T) = 9 ; n (L ∩ B) = 6 ; n (T ∩ B) = 4

n (L ∩ T ∩ B) = 8 ; n(L ∪ T ∪ B)= 35

n(L ∪ T ∪ B)= n(L) + n(T) + n(B) – n(L ∩ T) – n(L ∩ B) – n(T ∩ B) + n(L ∩ T ∩ B)

35 = 17 + 11 + n(B) - 9 - 6 - 4 + 8

35 = 17 + n(B)

n(B) = 18

11. A shopping mall has150 employees labelled 1 to 150, representing the Universal set U. The employees fall into the following categories:
    - Set A: 40 employees with a salary range of 30k-45k, labelled from 50 to 89.
    - Set B: 50 employees with a salary range of 50k-80k, labelled from 101 to 150.
    - Set C: 60 employees with a salary range of 100k-150k, labelled from 1 to 49 and 90 to 100.

    (a) Find $(A' \cup B') \cap C$ (a) Find $n\{A \cap (B^c \cap C^C)\}$

Solution

$U = \{1,2,3,\ldots,150\}$; $n(U) = 150$

$A = \{50,51,52,\ldots,89\}$; $n(A) = 40$

$B = \{101,102,\ldots,150\}$; $n(B) = 50$

$C = \{1,2,3,\ldots,49,90,91,\ldots,100\}$; $n(C) = 60$

$A' = U - A = \{1,2,3,\ldots,150\} - \{50,51,52,\ldots,89\} = \{1,2,3,\ldots,49,90,91,\ldots,100\}$
$B' = U - B = \{1,2,3,\ldots,150\} - \{101,102,\ldots,150\} = \{1,2,3,\ldots,100\}$
$C' = U - C = \{1,2,3,\ldots,150\} - \{1,2,3,\ldots,49,90,91,\ldots,100\} = \{50,51,52,\ldots,89\}$
(i) $(\mathbf{A'} \cup \mathbf{B'}) \cap \mathbf{C}$ =???
$A' \cup B' = \{1,2,3,\ldots,49,90,91,\ldots,100\} \cup \{1,2,3,\ldots,100\} = \{1,2,3,\ldots,100\}$
$(A' \cup B') \cap C = \{1,2,3,\ldots,100\} \cap \{1,2,3,\ldots,49,90,91,\ldots,100\}$
$(A' \cup B') \cap C = \{1,2,3,\ldots,49,90,91,\ldots,100\}$

(ii) $\boldsymbol{n\{A \cap (B' \cap C')\} = ??}$

$B' \cap C' = \{1,2,3, \dots ,100\} \cap \{50,51,52, \dots ,89\} = \{50,51,52, \dots ,89\}$

$A \cap (B' \cap C') = \{50,51,52, \dots ,89\} \cap \{50,51,52, \dots ,89\} = \{50,51,52, \dots ,89\}$

$\boldsymbol{n\{A \cap (B' \cap C')\} = 40}$

12. In a secondary school with 125 students participate in at least one of the following sports: cricket, football, or hockey.
    - 60 students play cricket.
    - 70 students play football.
    - 40 students play hockey.
    - 25 students play both cricket and football.
    - 15 students play both football and hockey.
    - 10 students play both cricket and hockey.

    (a) How many students play all three sports?

    (b) Draw a Venn diagram showing the distribution of sports participation in all the games.

Solution

n(C ∪ F ∪ H) = 125 ; n (C) = 60 ; n (F) = 70 ; n (H) = 40

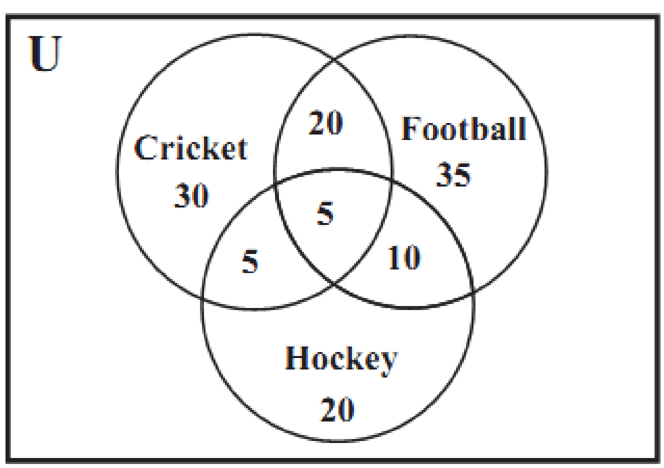
n (C ∩ F) = 25 ; n (F ∩ H) = 15 ; n (C ∩ H) = 10 ; n (C ∩ F ∩ H) = ???

n(C ∪ F ∪ H) = n(C) + n(F) + n(H) – n(C ∩ F) – n(F ∩ H) – n(C ∩ H) + n(C ∩ F ∩ H)

125 = 60 + 70 + 40 – 25 – 15 – 10 + n(C ∩ F ∩ H)

n(C ∩ F ∩ H) = 125 - 60 - 70 - 40 + 25 + 15 + 10

n(C ∩ F ∩ H) = 5



13. A survey was conducted in which 130 people were asked about their favourite foods. The survey results showed the following information:
    - 40 people said they liked nihari
    - 65 people said they liked biryani
    - 50 people said they liked korma
    - 20 people said they liked nihari and biryani
    - 35 people said they liked biryani and korma
    - 27 people said they liked nihari and korma
    - 12 people said they liked all three foods nihari, biryani, and korma

    (a) At least how many people like nihari, biryani or korma?

    (b) How many people did not like nihari, biryani, or korma?

    (c) How many people like only one of the following foods: nihari, biryani, or korma?

    (d) Draw a Venn diagram.

Solution

$n(N \cup B \cup K) = ???$; $n(N) = 40$; $n(B) = 65$; $n(K) = 50$

$n(N \cap B) = 20$; $n(B \cap K) = 35$; $n(N \cap K) = 27$; $n(N \cap B \cap K) = 12$

a) At least how many people like Nihari, Biryani or Korma:

$n(N \cup B \cup K) = n(N) + n(B) + n(K) - n(N \cap B) - n(B \cap K) - n(N \cap K) + n(N \cap B \cap K)$

$n(N \cup B \cup K) = 40 + 65 + 50 - 20 - 35 - 27 + 12$

$n(N \cup B \cup K) = \mathbf{85}$

b) How many people did not like Nihari, Biryani or Korma:

Total people = 130

People who like nihari, biryani, or korma = 85

People who did not like nihari, biryani, or korma = 130 – 85 = **45**

c) How many people like only one of the Nihari, Biryani or Korma:

People who like only nihari = $n(N) - n(N \cap B) - n(N \cap K) + n(N \cap B \cap K)$

$= 40 - 20 - 27 + 12 = 5$

People who like only biryani = n(B) – n(N ∩ B) – n(B ∩ K) + n(N ∩ B ∩ K)

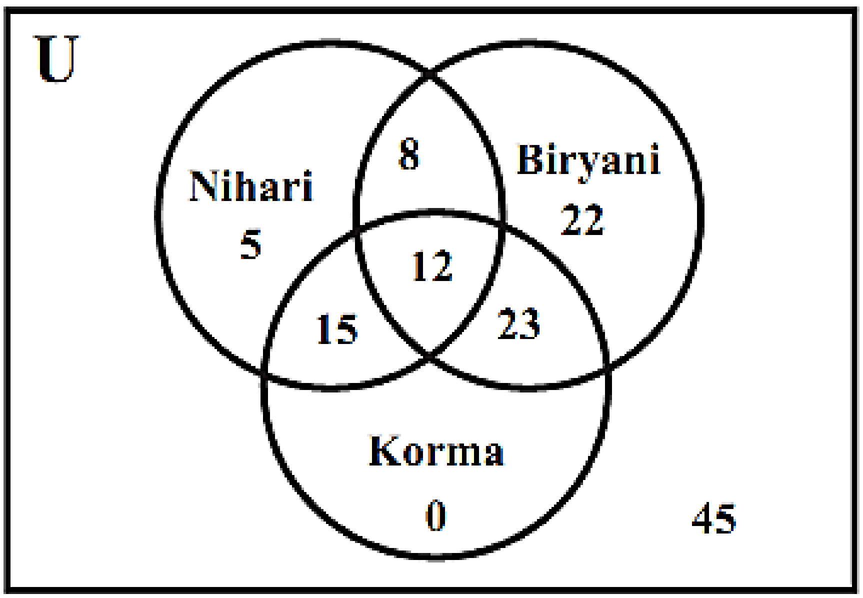
= 65 - 20 - 35 + 12 = 22

People who like only korma = n(K) – n(N ∩ K) – n(B ∩ K) + n(N ∩ B ∩ K)

= 50 - 27 - 35 + 12 = 0

Total people who like only one food = 5 + 22 + 0 = **27**

(d)



EXERCISE 3.3

1. For $A = \{1, 2, 3, 4\}$, find the following relations in A. State the domain and range of each relation.

(i) $\{(x, y) \mid y = x\}$ (ii) $\{(x, y) \mid y + x = 5\}$

(iii) $\{(x, y) \mid x + y < 5\}$ (iv) $\{(x, y) \mid x + y > 5\}$

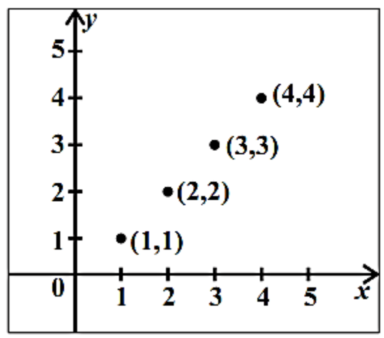
Solution

$A = \{1,2,3,4\}$

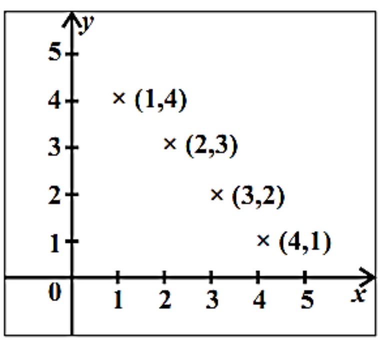
$A \times A = \{1,2,3,4\} \times \{1,2,3,4\} =$

$\{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$

(i) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
Domain of (i) $= \{1, 2, 3, 4\}$
Range of (i) $= \{1, 2, 3, 4\}$

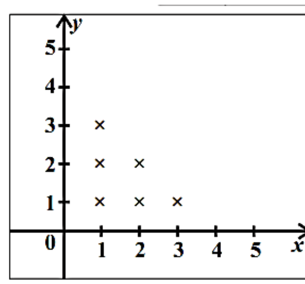


(ii)

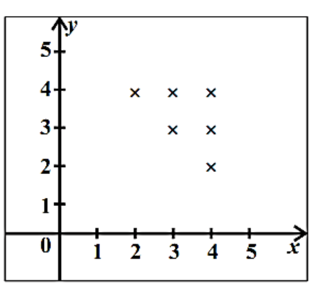


$\{(1, 4), (2, 3), (3, 2), (4, 1)\}$
Domain of (ii) $= \{1, 2, 3, 4\}$
Range of (ii) $= \{1, 2, 3, 4\}$

(iii) $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\}$
Domain of (iii) $= \{1, 2, 3\}$
Range of (iii) $= \{1, 2, 3\}$



(iv)



$\{(2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$
Domain of (iv) $= \{2, 3, 4\}$
Range of (iv) $= \{2, 3, 4\}$

2. Which of the following diagrams represent functions and of which type?

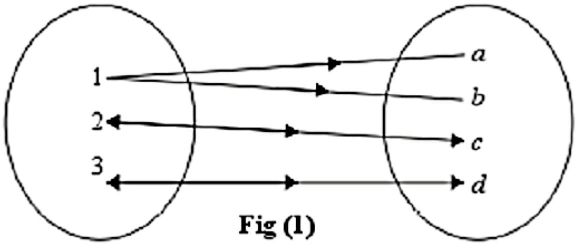


Fig (1)

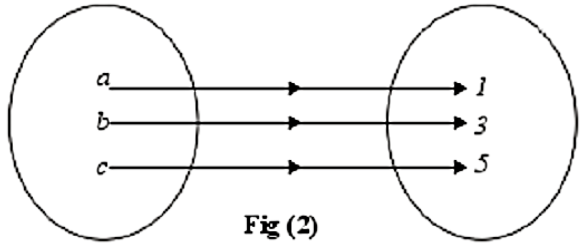


Fig (2)

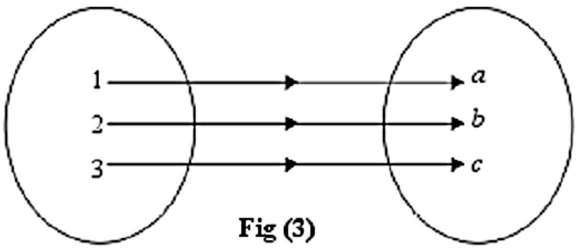


Fig (3)

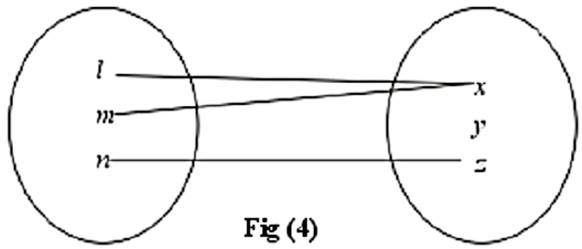


Fig (4)

Solution

Fig (1) does not represent a function. Fig (2) represents a function, which is a bijective function.
Fig (3) represents a function, which is a bijective function.
Fig (4) represents a function, which is an into function.

3. If $g(x) = 3x + 2$ and $h(x) = x^2 + 1$, then find:

(i) $g(0)$ (ii) $g(-3)$ (iii) $g\left(\frac{2}{3}\right)$

(iv) $h(1)$ (v) $h(-4)$ (vi) $h\left(-\frac{1}{2}\right)$

Solution

i. $g(x) = 3x + 2 \Rightarrow g(0) = 3(0) + 2 \Rightarrow g(0) = 2$

ii. $g(x) = 3x + 2 \Rightarrow g(-3) = 3(-3) + 2 \Rightarrow g(-3) = -9 + 2 \Rightarrow g(-3) = -7$

iii. $g(x) = 3x + 2 \Rightarrow g\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right) + 2 \Rightarrow g\left(\frac{2}{3}\right) = 2 + 2 \Rightarrow g\left(\frac{2}{3}\right) = 4$

iv. $h(x) = x^2 + 1 \Rightarrow h(1) = (1)^2 + 1 \Rightarrow h(1) = 1 + 1 \Rightarrow h(1) = 2$

v. $h(x) = x^2 + 1 \Rightarrow h(-4) = (-4)^2 + 1 \Rightarrow h(-4) = 16 + 1 \Rightarrow h(-4) = 17$

vi. $h(x) = x^2 + 1 \Rightarrow h\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 + 1 \Rightarrow h\left(-\frac{1}{2}\right) = \frac{1}{4} + 1 \Rightarrow h\left(-\frac{1}{2}\right) = \frac{5}{4}$

4. Given that $f(x) = ax + b + 1$, where a and b are constant numbers. If $f(3) = 8$ and $f(6) = 14$, then find the values of a and b.

Solution

$f(x) = ax + b + 1$

$f(3) = a(3) + b + 1$	$f(6) = a(6) + b + 1$
$8 = 3a + b + 1$	$14 = 6a + b + 1$
$8 - 1 = 3a + b$	$14 - 1 = 6a + b$
$\mathbf{3a + b = 7}$ **......(i)**	$\mathbf{6a + b = 13}$ **......(ii)**

2(i) - (ii)	**(ii) - (i)**
$6a + 2b = 14$	$6a + b = 13$
$-6a \pm b = -13$	$-3a \pm b = -7$
$\mathbf{b = 1}$	$\mathbf{a = 2}$

5. Given that $g(x) = ax + b + 5$, where a and b are constant numbers. If $g(-1) = 0$ and $g(2) = 10$, find the values of a and b.

Solution

$g(x) = ax + b + 5$

$g(-1) = a(-1) + b + 5$	$g(2) = a(2) + b + 5$
$0 = -a + b + 5$	$10 = 2a + b + 5$
$0 - 5 = -a + b$	$10 - 5 = 2a + b$
$\mathbf{-a + b = -5}$ **......(i)**	$\mathbf{2a + b = 5}$ **......(ii)**

(i) - (ii)	**2(i) + (ii)**
$-a + b = -5$	$-2a + 2b = -10$
$-2a \pm b = -5$	$2a + b = 5$
$\mathbf{a = \frac{10}{3}}$	$\mathbf{b = -\frac{5}{3}}$

6. Consider the function defined by $f(x) = 5x + 1$. If $f(x) = 32$, find the x value.

Solution

$f(x) = 5x + 1$	$f(x) = 5x + 1$
$f(x) = 32$	$\boldsymbol{f(x) = 31}$
$5x + 1 = 32$	$5x + 1 = 31$
$5x = 32 - 1$	$5x = 31 - 1$
$x = \frac{31}{5}$ **6 is wrong answer in book**	$x = \frac{30}{5} = \mathbf{6}$ **according to book**

7. Consider the function $f(x) = cx^2 + d$, where c and d are constant numbers. If $f(1) = 6$ and $f(-2) = 10$, then find the values of c and d.

Solution

$f(x) = cx^2 + d$

$f(1) = c(1)^2 + d$	$f(-2) = c(-2)^2 + d$
$\mathbf{c + d = 6}$ **......(i)**	$\mathbf{4c + d = 10}$ **......(ii)**

(i) - (ii)	**4(i) - (ii)**
$c + d = 6$	$4c + 4d = 24$
$-4c \pm d = -10$	$-4c \pm d = -10$
$\mathbf{c = \frac{4}{3}}$	$\mathbf{d = \frac{14}{3}}$

REVIEW EXERCISE 3

1. Four options are given against each statement. Encircle the correct option.

(i) The set builder form of the set $\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\right\}$ is:

(a) $\left\{x \mid x = \frac{1}{n}, n \in W\right\}$ (b) $\left\{x \mid x = \frac{1}{2n+1}, n \in W\right\}$

(c) $\left\{x \mid x = \frac{1}{n+1}, n \in W\right\}$ (d) $\{x \mid x = 2n+1, n \in W\}$

(ii) If $A = \{\ \}$, then $P(A)$ is:

(a) $\{\ \}$ (b) $\{1\}$ (c) $\{\{\ \}\}$ (d) ϕ

(iii) If $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $U - (A \cap B)$ is:

(a) $\{1, 2, 4, 5\}$ (b) $\{2, 3\}$ (c) $\{1, 3, 4, 5\}$ (d) $\{1, 2, 3\}$

(iv) If A and B are overlapping sets, then $n(A - B)$ is equal to

(a) $n(A)$ (b) $n(B)$ (c) $A \cap B$ (d) $n(A) - n(A \cap B)$

(v) If $A \subseteq B$ and $B - A \neq \phi$, then $n(B - A)$ is equal to

(a) 0 (b) $n(B)$ (c) $n(A)$ (d) $n(B) - n(A)$

(vi) If $n(A \cup B) = 50$, $n(A) = 30$ and $n(B) = 35$, then $n(A \cap B) =$:

(a) 23 (b) 15 (c) 9 (d) 40

(vii) If $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$, then cartesian product of A and B contains exactly ________ elements.

(a) 13 (b) 12 (c) 10 (d) 6

(viii) If $f(x) = x^2 - 3x + 2$, then the value of $f(a + 1)$ is equal to:

(a) $a + 1$ (b) $a^2 + 1$ (c) $a^2 + 2a + 1$ (d) $a^2 - a$

(ix) Given that $f(x) = 3x+1$, if $f(x)=28$, then the value of x is:

(a) 9 (b) 27 (c) 3 (d) 18

(x) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$ two non-empty sets and $f : A \to B$ be a function defined as $f = \{(1, a), (2, b), (3, b)\}$, then which of the following statement is true?

(a) f is injective (b) f is surjective (c) f is bijective (d) f is into only

2. Write each of the following sets in tabular forms:

(i) $\{x|x=2n, n\in N\}$ (ii) $\{x|x=2m+1, m\in N\}$

(iii) $\{x|x=11n, n\in W \wedge n<11\}$ (iv) $\{x|x\in E \wedge 4<x<6\}$

(v) $\{x|x\in O \wedge 5\le x<7\}$ (vi) $\{x|x\in Q \wedge x^2=2\}$

(vii) $\{x|x\in Q \wedge x=-x\}$ (viii) $\{x|\, x\in R \wedge x\notin Q'\}$

Solution

(i) $\{2, 4, 6, 8, 10, \ldots\}$ (ii) $\{3, 5, 7, 9, 11, \ldots\}$

(iii) $\{0, 11, 22, 33, 44, 55, 66, 77, 88, 99, 110\}$

(iv) ϕ (v) ϕ (vi) ϕ

(vii) $\{0\}$ (viii) Q

3. Let $U=\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A=\{2, 4, 6, 8, 10\}$, $B=\{1, 2, 3, 4, 5\}$ and $C=\{1, 3, 5, 7, 9\}$

List the members of each of the following sets:

(i) A' (ii) B' (iii) $A\cup B$ (iv) $A-B$

(v) $A\cap C$ (vi) $A'\cup C'$ (vii) $A'\cup C$ (viii) U'

Solution

i. $A' = U - A = \{1,2,3,4,5,6,7,8,9,10\} - \{2,4,6,8,10\} = \{1,3,5,7,9\}$

ii. $B' = U - B = \{1,2,3,4,5,6,7,8,9,10\} - \{1,2,3,4,5\} = \{6,7,8,9,10\}$

iii. $A \cup B = \{2,4,6,8,10\} \cup \{1,2,3,4,5\} = \{1,2,3,4,5,6,7,8,9,10\}$

iv. $A - B = \{2,4,6,8,10\} - \{1,2,3,4,5\} = \{6,8,10\}$

v. $A \cap C = \{2,4,6,8,10\} \cap \{1,3,5,7,9\} = \varphi$

vi. $A' \cup C' = \{1,3,5,7,9\} \cup \{2,4,6,8,10\} = \{1,2,3,4,5,6,7,8,9,10\}$

vii. $A' \cup C = \{1,3,5,7,9\} \cup \{1,3,5,7,9\} = \{1,3,5,7,9\}$

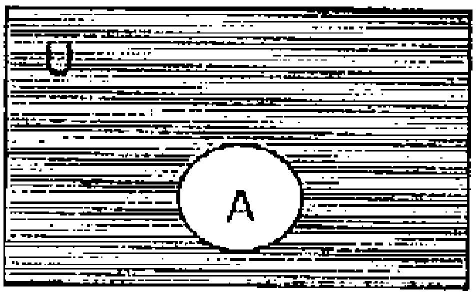
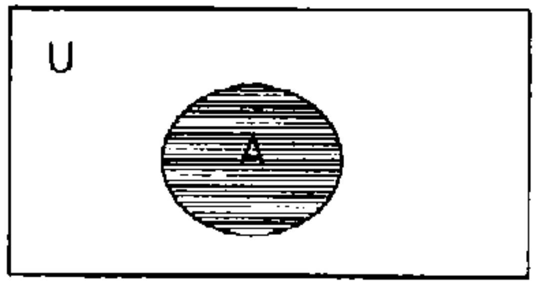
viii. $U' = U - U = \{1,2,3,4,5,6,7,8,9,10\} - \{1,2,3,4,5,6,7,8,9,10\} = \varphi$

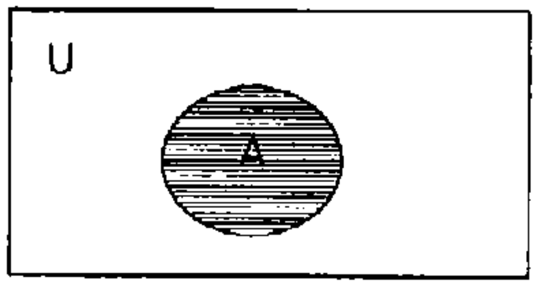
4. Using the Venn diagrams, if necessary, find the single sets equal to the following:

(i) A' (ii) $A \cap U$ (iii) $A \cup U$

(iv) $A \cup \phi$ (v) $\phi \cap \phi$

Solution

<table>
<tr><td>(i) $A' = U - A$

</td><td>(ii) $A \cap U = A$

</td></tr>
<tr><td>(iii) $A \cup U = U$

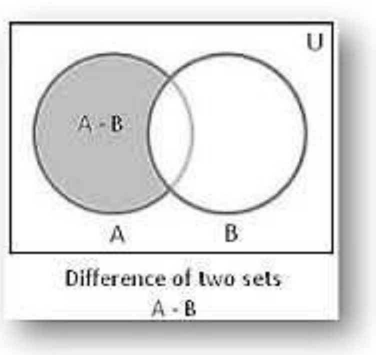
</td><td>(iv) $A \cup \varphi = A$

</td></tr>
<tr><td colspan="2">(v) $\varphi \cap \varphi = \varphi$
It has no Venn diagram.</td></tr>
</table>

5. Use Venn diagrams to verify the following:

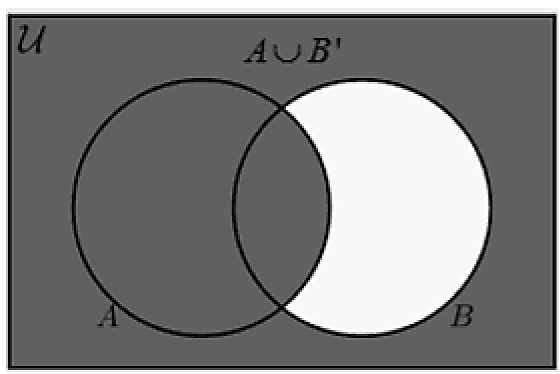
(i) $A - B = A \cup B'$ (ii) $(A - B)' \cap B = B$

Solution

(i) $A - B = A \cup B'$



not equals to



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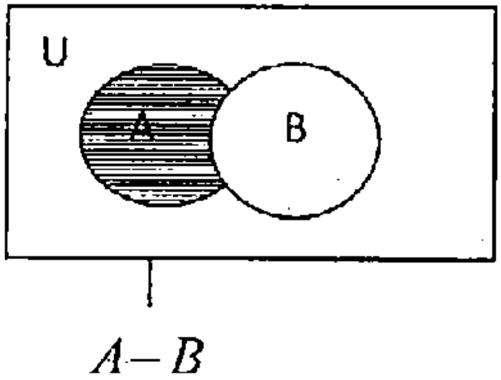
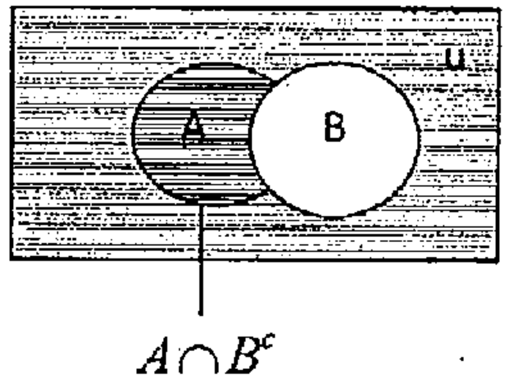
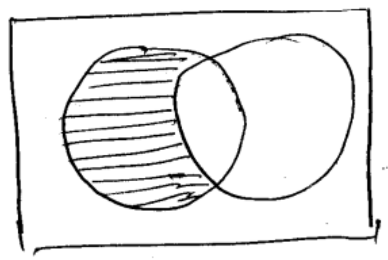
(i) $\boldsymbol{A - B = A \cap B'}$



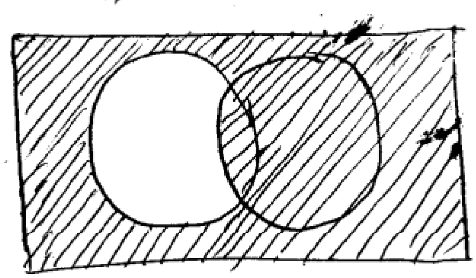


(ii) $\boldsymbol{(A - B)' \cap B = B}$

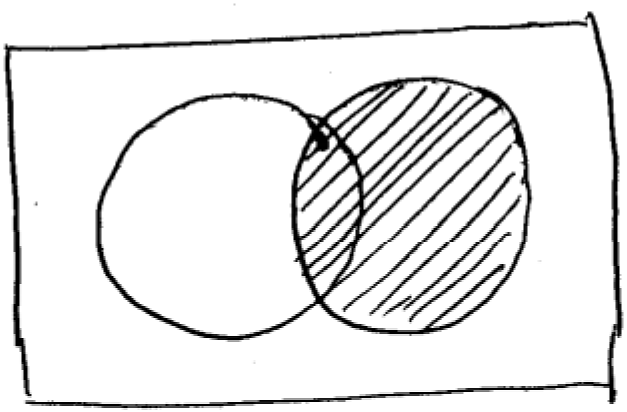
$A - B$



$(A - B)'$



Then $\boldsymbol{(A - B)' \cap B = B}$ **is**



6. Verify the properties for the sets A, B and C given below:
   (i) Associativity of Union (ii) Associativity of intersection.
   (iii) Distributivity of Union over intersection.
   (iv) Distributivity of intersection over union.
   (a) $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7, 8\}$, $C = \{5, 6, 7, 9, 10\}$
   (b) $A = \phi$, $B = \{0\}$, $C = \{0, 1, 2\}$
   (c) $A = N$, $B = Z$, $C = Q$

Solution

(a) $A = \{1,2,3,4\}\ ; B = \{3,4,5,6,7,8\}\ ; C = \{5,6,7,9,10\}$

i. Associativity of Union: $\mathbf{A \cup (B \cup C) = (A \cup B) \cup C}$

$L.H.S = A \cup (B \cup C) = \{1,2,3,4\} \cup [\{3,4,5,6,7,8\} \cup \{5,6,7,9,10\}]$

$A \cup (B \cup C) = \{1,2,3,4\} \cup \{3,4,5,6,7,8,9,10\} = \{1,2,3,4,5,6,7,8,9,10\}$

$R.H.S = (A \cup B) \cup C = [\{1,2,3,4\} \cup \{3,4,5,6,7,8\}] \cup \{5,6,7,9,10\}$

$(A \cup B) \cup C = \{1,2,3,4,5,6,7,8\} \cup \{5,6,7,9,10\} = \{1,2,3,4,5,6,7,8,9,10\}$

Hence $A \cup (B \cup C) = (A \cup B) \cup C$

ii. Associativity of Intersection: $\mathbf{A \cap (B \cap C) = (A \cap B) \cap C}$

$L.H.S = A \cap (B \cap C) = \{1,2,3,4\} \cap [\{3,4,5,6,7,8\} \cap \{5,6,7,9,10\}]$

$A \cap (B \cap C) = \{1,2,3,4\} \cap \{5,6,7\} = \{\ \ \}$

$R.H.S = (A \cap B) \cap C = [\{1,2,3,4\} \cap \{3,4,5,6,7,8\}] \cap \{5,6,7,9,10\}$

$(A \cap B) \cap C = \{3,4\} \cap \{5,6,7,9,10\} = \{\ \ \}$

Hence $A \cap (B \cap C) = (A \cap B) \cap C$

iii. Distributivity of Union over Intersection: $\mathbf{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$

$L.H.S = A \cup (B \cap C) = \{1,2,3,4\} \cup [\{3,4,5,6,7,8\} \cap \{5,6,7,9,10\}]$

$A \cup (B \cap C) = \{1,2,3,4\} \cup \{5,6,7\} = \{1,2,3,4,5,6,7\}$

$R.H.S = (A \cup B) \cap (A \cup C) = [\{1,2,3,4\} \cup \{3,4,5,6,7,8\}] \cap [\{1,2,3,4\} \cup \{5,6,7,9,10\}]$

$(A \cup B) \cap (A \cup C) = \{1,2,3,4,5,6,7,8\} \cap \{1,2,3,4,5,6,7,9,10\} = \{1,2,3,4,5,6,7\}$

Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

iv. Distributivity of Intersection over Union: $\mathbf{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

$L.H.S = A \cap (B \cup C) = \{1,2,3,4\} \cap [\{3,4,5,6,7,8\} \cup \{5,6,7,9,10\}]$

$A \cap (B \cup C) = \{1,2,3,4\} \cap \{3,4,5,6,7,8,9,10\} = \{3,4\}$

$R.H.S = (A \cap B) \cup (A \cap C) = [\{1,2,3,4\} \cap \{3,4,5,6,7,8\}] \cup [\{1,2,3,4\} \cap \{5,6,7,9,10\}]$

$(A \cap B) \cup (A \cap C) = \{3,4\} \cup \{\ \ \} = \{3,4\}$

Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) $A = \varphi\,; B = \{0\}\,; C = \{0,1,2\}$

i. Associativity of Union: $\mathbf{A} \cup (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cup \mathbf{C}$

$\text{L.H.S} = A \cup (B \cup C) = \varphi \cup [\{0\} \cup \{0,1,2\}] = \varphi \cup \{0,1,2\} = \{0,1,2\}$

$\text{R.H.S} = (A \cup B) \cup C = [\varphi \cup \{0\}] \cup \{0,1,2\} = \{0\} \cup \{0,1,2\} = \{0,1,2\}$

Hence $A \cup (B \cup C) = (A \cup B) \cup C$

ii. Associativity of Intersection: $\mathbf{A} \cap (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C}$

$\text{L.H.S} = A \cap (B \cap C) = \varphi \cap [\{0\} \cap \{0,1,2\}] = \varphi \cap \{0\} = \varphi$

$\text{R.H.S} = (A \cap B) \cap C = [\varphi \cap \{0\}] \cap \{0,1,2\} = \varphi \cap \{0,1,2\} = \varphi$

Hence $A \cap (B \cap C) = (A \cap B) \cap C$

iii. Distributivity of Union over Intersection: $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$

$\text{L.H.S} = A \cup (B \cap C) = \varphi \cup [\{0\} \cap \{0,1,2\}] = \varphi \cup \{0\} = \{0\}$

$\text{R.H.S} = (A \cup B) \cap (A \cup C) = [\varphi \cup \{0\}] \cap [\varphi \cup \{0,1,2\}] = \{0\} \cap \{0,1,2\} = \{0\}$

Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

iv. Distributivity of Intersection over Union: $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$

$\text{L.H.S} = A \cap (B \cup C) = \varphi \cap [\{0\} \cup \{0,1,2\}] = \varphi \cap \{0,1,2\} = \varphi$

$\text{R.H.S} = (A \cap B) \cup (A \cap C) = [\varphi \cap \{0\}] \cup [\varphi \cap \{0,1,2\}] = \varphi \cup \varphi = \varphi$

Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(c) $A = N = \{1,2,3,\dots\}\,; B = Z = \{0, \pm 1, \pm 2, \pm 3, \dots\}\,; C = Q\,;\ \mathbf{N} \leq \mathbf{Z} \leq \mathbf{Q}$

i. Associativity of Union: $\mathbf{A} \cup (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cup \mathbf{C}$

$\text{L.H.S} = A \cup (B \cup C) = N \cup [Z \cup Q] = N \cup Q = Q$

$\text{R.H.S} = (A \cup B) \cup C = [N \cup Z] \cup Q = Z \cup Q = Q$

Hence $A \cup (B \cup C) = (A \cup B) \cup C$

ii. Associativity of Intersection: $\mathbf{A} \cap (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C}$

$\text{L.H.S} = A \cap (B \cap C) = N \cap [Z \cap Q] = N \cap Z = N$

$\text{R.H.S} = (A \cap B) \cap C = [N \cap Z] \cap Q = N \cap Q = N$

Hence $A \cap (B \cap C) = (A \cap B) \cap C$

iii. Distributivity of Union over Intersection: $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$

$\text{L.H.S} = A \cup (B \cap C) = N \cup [Z \cap Q] = N \cup Z = Z$

$\text{R.H.S} = (A \cup B) \cap (A \cup C) = [N \cup Z] \cap [N \cup Q] = Z \cap Q = Z$

Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

iv. Distributivity of Intersection over Union: $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$

$\text{L.H.S} = A \cap (B \cup C) = N \cap [Z \cup Q] = N \cap Q = N$

$\text{R.H.S} = (A \cap B) \cup (A \cap C) = [N \cap Z] \cup [N \cap Q] = N \cup N = N$

Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

7. Verify De Morgan's Laws for the following sets:
$U = \{1, 2, 3, \ldots, 20\}$, $A = \{2, 4, 6, \ldots, 20\}$ and $B = \{1, 3, 5, \ldots, 19\}$.

Solution

$(\mathbf{A} \cup \mathbf{B})' = \mathbf{A}' \cap \mathbf{B}'$ $U = \{1, 2, 3, ..., 20\}$ $A = \{2, 4, 6, ..., 20\}$ $B = \{1, 3, 5, ..., 19\}$ $A' = U - A$ $= \{1, 2, 3, ..., 20\} - \{2, 4, 6, ..., 20\}$ $= \{1, 3, 5, ..., 19\}$ $B' = U - B$ $= \{1, 2, 3, ..., 20\} - \{1, 3, 5, ..., 19\}$ $= \{2, 4, 6, ..., 20\}$ $A \cup B = \{1, 2, 3, ..., 20\}$ $(A \cup B)' = U - (A \cup B)$ $= \{1, 2, 3, ..., 20\} - \{1, 2, 3, ..., 20\}$ $= \{\ \}$ $A' \cap B' = \{1, 3, 5, ..., 19\} \cap \{2, 4, 6, ..., 20\}$ $= \{\ \}$ Hence $(A \cup B)' = A' \cap B'$	$(\mathbf{A} \cap \mathbf{B})' = \mathbf{A}' \cup \mathbf{B}'$ $U = \{1, 2, 3, ..., 20\}$ $A = \{2, 4, 6, ..., 20\}$ $B = \{1, 3, 5, ..., 19\}$ $A' = U - A$ $= \{1, 2, 3, ..., 20\} - \{2, 4, 6, ..., 20\}$ $= \{1, 3, 5, ..., 19\}$ $B' = U - B$ $= \{1, 2, 3, ..., 20\} - \{1, 3, 5, ..., 19\}$ $= \{2, 4, 6, ..., 20\}$ $A \cap B = \{\ \}$ $(A \cap B)' = U - (A \cap B)$ $= \{1, 2, 3, ..., 20\} - \{\ \}$ $= \{1, 2, 3, ..., 20\}$ $A' \cup B' = \{1, 3, 5, ..., 19\} \cup \{2, 4, 6, ..., 20\}$ $= \{1, 2, 3, ..., 20\}$ Hence $(A \cap B)' = A' \cup B'$

8. Consider the set $P = \{x \mid x = 5m, m \in N\}$ and $Q = \{x \mid x = 2m, m \in N\}$. Find $P \cap Q$

Solution

$P = \{x \mid x = 5m, m \in N\} = \{5, 10, 15, 20, 25, ...\}$

$Q = \{x \mid x = 2m, m \in N\} = \{2, 4, 6, 8, 10, 12, ...\}$

$P \cap Q = \{5, 10, 15, 20, 25, ...\} \cap \{2, 4, 6, 8, 10, 12, ...\}$

$P \cap Q = \{10, 20, 30, 40, 50, ...\} = \{x \mid x = 10m, m \in N\}$

9. From suitable properties of union and intersection, deduce the following results:
(i) $A \cap (A \cup B) = A \cup (A \cap B)$ (ii) $A \cup (A \cap B) = A \cap (A \cup B)$

Solution

(i) L.H.S. = $A \cap (A \cup B) = (A \cap A) \cup (A \cap B) = A \cup (A \cap B) =$ **R.H.S.**

(i) L.H.S. = $A \cup (A \cap B) = (A \cup A) \cap (A \cup B) = A \cap (A \cup B) =$ **R.H.S.**

10. If $g(x) = 7x - 2$ and $s(x) = 8x^2 - 3$ find:

(i) $g(0)$ (ii) $g(-1)$ (iii) $g\left(-\frac{5}{3}\right)$ (iv) $s(1)$ (v) $s(-9)$ (vi) $s\left(\frac{7}{2}\right)$

Solution

i. $g(x) = 7x - 2 \Rightarrow g(0) = 7(0) - 2 \Rightarrow g(0) = -2$

ii. $g(x) = 7x - 2 \Rightarrow g(-1) = 7(-1) - 2 \Rightarrow g(-1) = -7 - 2 \Rightarrow g(-1) = -9$

iii. $g(x) = 7x - 2 \Rightarrow g\left(-\frac{5}{3}\right) = 7\left(-\frac{5}{3}\right) - 2 \Rightarrow g\left(-\frac{5}{3}\right) = -\frac{35}{3} - 2 \Rightarrow g\left(-\frac{5}{3}\right) = -\frac{41}{3}$

iv. $s(x) = 8x^2 - 3 \Rightarrow s(1) = 8(1)^2 - 3 \Rightarrow s(1) = 8 - 3 \Rightarrow s(1) = 5$

v. $s(x) = 8x^2 - 3 \Rightarrow s(-9) = 8(-9)^2 - 3 \Rightarrow s(-9) = 648 - 3 \Rightarrow s(-9) = 645$

vi. $s(x) = 8x^2 - 3 \Rightarrow s\left(\frac{7}{2}\right) = 8\left(\frac{7}{2}\right)^2 - 3 \Rightarrow s\left(\frac{7}{2}\right) = 98 - 3 \Rightarrow s\left(\frac{7}{2}\right) = 95$

11. Given that $f(x) = ax + b$, where a and b are constant numbers. If $f(-2) = 3$ and $f(4) = 10$, then find the values of a and b.

Solution

$f(x) = ax + b$

$f(-2) = a(-2) + b$

$\mathbf{-2a + b = 3}$ **……(i)**

$f(4) = a(4) + b$

$\mathbf{4a + b = 10}$ **……(ii)**

(i) - (ii)

$$\begin{aligned} -2a + b &= 3 \\ -4a \pm b &= -10 \\ \hline \end{aligned}$$

$$\mathbf{a = \frac{7}{6}}$$

2(i) + (ii)

$$\begin{aligned} -4a + 2b &= 6 \\ 4a + b &= 10 \\ \hline \end{aligned}$$

$$\mathbf{b = \frac{16}{3}}$$

12. Consider the function defined by $k(x) = 7x - 5$. If $k(x) = 100$, find the value of x.

Solution

$k(x) = 7x - 5$

Using $k(x) = 100$

$7x - 5 = 100$

$7x = 100 + 5$

$x = \frac{105}{7}$

13. Consider the function $g(x) = mx^2+n$, where m and n are constant numbers. If $g(4) = 20$ and $g(0) = 5$, find the values of m and n.

Solution

$g(x) = mx^2 + n$

$g(0) = m(0)^2 + n$

$\mathbf{n = 5}$

Now $g(4) = m(4)^2 + n$

$\mathbf{16m + n = 20}$

Using $n = 5$

$16m + 5 = 20$

$\mathbf{m = \frac{15}{16}}$

14. A shopping mall has 100 products from various categories labeled 1 to 100, representing the universal set U. The products are categorized as follows:
    - Set A: Electronics, consisting of 30 products labeled from 1 to 30.
    - Set B: Clothing comprises 25 products labeled from 31 to 55.
    - Set C: Beauty Products, comprising 25 products labeled from 76 to 100.

    Write each set in tabular form, and find the union of all three sets.

Solution

U = {1, 2, 3, ... , 100}

A = {1, 2, 3, ... , 30}

B = {31, 32, 33, ... , 55}

C = {76, 77, 78, ... , 100}

A ∪ B ∪ C = {1, 2, 3, ..., 30} ∪ {31, 32, ..., 55} ∪ {76, 77, ..., 100}

A ∪ B ∪ C = {1, 2, 3, ..., 30, 31, 32, ..., 55, 76, 77, ..., 100}

15. Out of the 180 students who appeared in the annual examination, 120 passed the math test, 90 passed the science test, and 60 passed both the math and science tests.
    (a) How many passed either the math or science test?
    (b) How many did not pass either of the two tests?
    (c) How many passed the science test but not the math test?
    (d) How many failed the science test?

Solution

Total students = 180

Passed Math = 120, Passed Science = 90, Passed both Math and Science = 60

1. How many passed either the Math or Science test?

Passed either Math or Science = Passed Math + Passed Science - Passed both

= 120 + 90 – 60 = 150

2. How many did not pass either of the two tests?

Failed both Math and Science = Total students - Passed either Math or Science

= 180 – 150 = 30

3. How many passed the Science test but not the Math test?

Passed Science but not Math = Passed Science - Passed both

= 90 – 60 = 30

4. How many failed the Science test?

Failed Science = Total students - Passed Science

= 180 – 90 = 90

16. In a software house of a city with 300 software developers, a survey was conducted to determine which programming languages are liked more. The survey revealed the following statistics:

- 150 developers like Python.
- 130 developers like Java.
- 120 developers like PHP.
- 70 developers like both Python and Java.
- 60 developers like both Python and PHP.
- 50 developers like both Java and PHP.
- 40 developers like all three languages: Python, Java and PHP.

  (a) How many developers use at least one of these languages?

  (b) How many developers use only one of these languages?

  (c) How many developers do not use any of these languages?

  (d) How many developers use only PHP?

Solution

Total developers = 300

$n(P) = 150, n(J) = 130, n(H) = 120, n(P \cap J) = 70, n(P \cap H) = 60, n(J \cap H) = 50$

$n(P \cap J \cap H) = 40$

1. How many developers use at least one of these languages?

$n(P \cup J \cup H) = n(P) + n(J) + n(H) – n(P \cap J) – n(P \cap H) – n(J \cap H) + n(P \cap J \cap H)$

$= 150 + 130 + 120 - 70 - 60 - 50 + 40 = \mathbf{260}$

2. How many developers use only one of these languages?

Developers who use only P $= n(P) – n(P \cap J) – n(P \cap H) + n(P \cap J \cap H)$

$= 150 - 70 - 60 + 40 = 60$

Developers who use only J = n(J) – n(P ∩ J) – n(J ∩ H) + n(P ∩ J ∩ H)

= 130 - 70 - 50 + 40 = 50

Developers who use only H = n(H) – n(P ∩ H) – n(J ∩ H) + n(P ∩ J ∩ H)

= 120 - 60 - 50 + 40 = 50

Total developers who use only one language = 60 + 50 + 50 = **160**

3. How many developers do not use any of these languages?

Developers who do not use any language = Total developers - Developers who use at least one language

= 300 – 260 = **40**

4. How many developers use only PHP?

Developers who use only PHP = n(H) = **50**

Unit 4 Factorization and Algebraic Manipulation

EXERCISE 4.1

1. Factorize by identifying common factors.

(i) $6x + 12$ (ii) $15y^2 + 20y$ (iii) $-12x^2 - 3x$

(iv) $4a^2b + 8ab^2$ (v) $xy - 3x^2 + 2x$ (vi) $3a^2b - 9ab^2 + 15ab$

Solution:

(i) $6(x + 2)$ (ii) $5y(3y + 4)$ (iii) $-3x(4x + 1)$

(iv) $4ab(a + 2b)$ (v) $x(y - 3x + 2)$

(vi) $3ab(a - 3b + 5)$

2. Factorize and represent pictorially:

(i) $5x + 15$ (ii) $x^2 + 4x + 3$ (iii) $x^2 + 6x + 8$

(iv) $x^2 + 4x + 4$

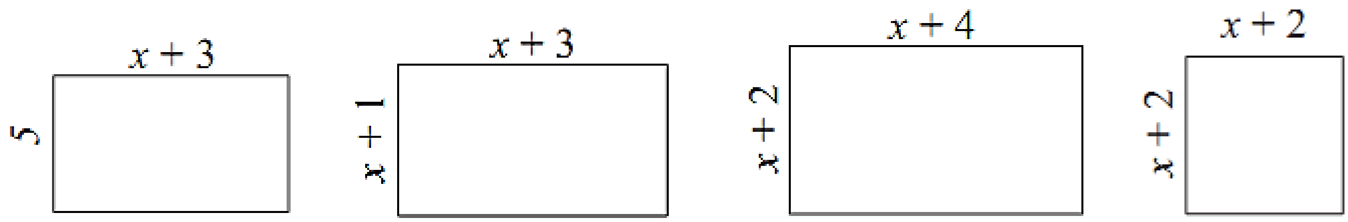
Solution

(i) $5(x+3)$

(ii) $x^2 + 4x + 3 = x^2 + 3x + x + 3 = x(x + 3) + 1(x + 3) = (x + 1)(x + 3)$

(iii) $x^2 + 6x + 8 = x^2 + 4x + 2x + 8 = x(x + 4) + 2(x + 4) = (x + 2)(x + 4)$

(iv) $x^2 + 4x + 4 = x^2 + 2x + 2x + 4 = x(x + 2) + 2(x + 2) = (x + 2)(x + 2) = (x + 2)^2$



3. Factorize:

(i)	x^2+x-12	(ii)	$x^2+7x+10$	(iii)	x^2-6x+8
(iv)	x^2-x-56	(v)	$x^2-10x-24$	(vi)	$y^2+4y-12$
(vii)	$y^2+13y+36$	(viii)	x^2-x-2		

Solution

(i) $x^2+x-12=x^2+4x-3x-12=x(x+4)-3(x+4)=(x+4)(x-3)$

(ii) $x^2+7x+10=x^2+5x+2x+10=x(x+5)+2(x+5)=(x+5)(x+2)$

(iii) $x^2-6x+8=x^2-4x-2x+8=x(x-4)-2(x-4)=(x-4)(x-2)$

(iv) $x^2-x-56=x^2-8x+7x-56=x(x-8)+7(x-8)=(x-8)(x+7)$

(v) $x^2-10x-24=x^2-12x+2x-24=x(x-12)+2(x-12)=(x-12)(x+2)$

(vi) $y^2+4y-12=y^2+6y-2y-12=y(y+6)-2(y+6)=(y+6)(y-2)$

(vii) $y^2+13y+36=y^2+9y+4y+36=y(y+9)+4(y+9)=(y+9)(y+4)$

(viii) $x^2-x-2=x^2-2x+x-2=x(x-2)+1(x-2)=(x-2)(x+1)$

4. Factorize:

(i)	$2x^2+7x+3$	(ii)	$2x^2+11x+15$	(iii)	$4x^2+13x+3$
(iv)	$3x^2+5x+2$	(v)	$3y^2-11y+6$	(vi)	$2y^2-5y+2$
(vii)	$4z^2-11z+6$	(viii)	$6+7x-3x^2$		

Solution

(i) $2x^2+7x+3=2x^2+6x+x+3=2x(x+3)+1(x+3)=(2x+1)(x+3)$

(ii) $2x^2+11x+15=2x^2+6x+5x+15=2x(x+3)+5(x+3)=(2x+5)(x+3)$

(iii) $4x^2+13x+3=4x^2+12x+x+3=4x(x+3)+1(x+3)=(4x+1)(x+3)$

(iv) $3x^2+5x+2=3x^2+3x+2x+2=3x(x+1)+2(x+1)=(3x+2)(x+1)$

(v) $3y^2-11y+6=3y^2-9y-2y+6=3y(y-3)-2(y-3)=(3y-2)(y-3)$

(vi) $2y^2-5y+2=2y^2-4y-y+2=2y(y-2)-1(y-2)=(2y-1)(y-2)$

(vii) $4z^2-11z+6=4z^2-8z-3z+6=4z(z-2)-3(z-2)=(4z-3)(z-2)$

(viii) $6+7x-3x^2=-3x^2+7x+6=-3x^2+9x-2x+6$

$=3x(-x+3)+2(-x+3)=(3x+2)(3-x)$

EXERCISE 4.2

1. Factorize each of the following expressions:

(i)	$4x^4 + 81y^4$	(ii)	$a^4 + 64b^4$	(iii)	$x^4 + 4x^2 + 16$
(iv)	$x^4 - 14x^2 + 1$	(v)	$x^4 - 30x^2y^2 + 9y^4$	(vi)	$x^4 - 11x^2y^2 + y^4$

Solution

1.(i) $4x^4 + 81y^4$

$= (2x^2)^2 + (9y^2)^2 = (2x^2)^2 + (9y^2)^2 + 2(2x^2)(9y^2) - 2(2x^2)(9y^2)$

$= (2x^2 + 9y^2)^2 - 36x^2y^2 = (2x^2 + 9y^2)^2 - (6xy)^2$

$= (2x^2 + 9y^2 - 6xy)(2x^2 + 9y^2 + 6xy)$

1.(ii) $a^4 + 64b^4$

$= (a^2)^2 + (8b^2)^2 = (a^2)^2 + (8b^2)^2 + 2(a^2)(8b^2) - 2(a^2)(8b^2)$

$= (a^2 + 8b^2)^2 - 16\,a^2b^2 = (a^2 + 8b^2)^2 - (4ab)^2$

$= (a^2 + 8b^2 + 4ab)(a^2 + 8b^2 - 4ab)$

1.(iii) $x^4 + 4x^2 + 16$

$= x^4 + 8x^2 - 4x^2 + 16 = x^4 + 8x^2 + 16 - 4x^2$

$= (x^2 + 4)^2 - (2x)^2$

$= (x^2 + 4 - 2x)(x^2 + 4 + 2x)$

1.(iv) $x^4 - 14x^2 + 1$

$= x^4 - 16x^2 + 2x^2 + 1 = x^4 + 2x^2 + 1 - 16x^2$

$= (x^2 + 1)^2 - (4x)^2$

$= (x^2 + 1 - 4x)(x^2 + 1 + 4x)$

1.(v) $\boldsymbol{x^4 - 30x^2y^2 + 9y^4}$

$= x^4 - 36x^2y^2 + 6x^2y^2 + 9y^4 = x^4 + 6x^2y^2 + 9y^4 - 36x^2y^2$

$= (x^2 + 3y^2)^2 - (6xy)^2$

$= (x^2 - 6xy + 3y^2)(x^2 + 6xy + 3y^2)$

1.(vi) $\boldsymbol{x^4 - 11x^2y^2 + y^4}$

$= x^4 - 9x^2y^2 + 2x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - 9x^2y^2$

$= (x^2 + y^2)^2 - (3xy)^2$

$= (x^2 - 3xy + y^2)(x^2 + 3xy + y^2)$

2. Factorize each of the following expressions:

(i) $(x+1)(x+2)(x+3)(x+4)+1$ (ii) $(x+2)(x-7)(x-4)(x-1)+17$

(iii) $(2x^2+7x+3)(2x^2+7x+5)+1$ (iv) $(3x^2+5x+3)(3x^2+5x+5)-3$

(v) $(x+1)(x+2)(x+3)(x+6)-3x^2$ (vi) $(x+1)(x-1)(x+2)(x-2)+13x^2$

Solution

2.(i) $\boldsymbol{(x+1)(x+2)(x+3)(x+4)+1}$

$= (x+1)(x+4)(x+2)(x+3)+1$

$= (x^2+5x+4)(x^2+5x+6)+1$

$= (y+4)(y+6)+1 = y^2+10y+24+1 = y^2+10y+25$

$= (y+5)^2 = (x^2+5x+5)^2$

2.(ii) $\boldsymbol{(x+2)(x-7)(x-4)(x-1)+17}$

$= (x+2)(x-7)(x-4)(x-1)+17$

$= (x^2-5x-14)(x^2-5x+4)+17$

$= (y-14)(y+4)+17 = y^2-10y-56+17 = y^2-10y-39$

$= y^2-13y+3y-39 = y(y-13)+3(y-13)$

$= (y-13)(y+3) = (x^2-5x-13)(x^2-5x+3)$

2.(iii) $(\mathbf{2x^2+7x+3})(\mathbf{2x^2+7x+5})+\mathbf{1}$

$=(2x^2+7x+3)(2x^2+7x+5)+1$

$=(y+3)(y+5)+1=y^2+8y+15+1=y^2+8y+16$

$=(y+4)^2=(2x^2+7x+4)^2$

2.(iv) $(\mathbf{3x^2+5x+3})(\mathbf{3x^2+5x+5})-\mathbf{3}$

$=(3x^2+5x+3)(3x^2+5x+5)-3$

$=(y+3)(y+5)-3=y^2+8y+15-3=y^2+8y+12$

$=y^2+6y+2y+12=y(y+6)+2(y+6)$

$=(y+6)(y+2)=(3x^2+5x+6)(3x^2+5x+2)$

2.(v) $(\mathbf{x+1})(\mathbf{x+2})(\mathbf{x+3})(\mathbf{x+6})-\mathbf{3x^2}$

$=(x+1)(x+6)(x+2)(x+3)-3x^2$

$=(x^2+7x+6)(x^2+5x+6)-3x^2$

$=(x^2+6+7x)(x^2+6+5x)-3x^2$

$=(y+7x)(y+5x)-3x^2=y^2+12xy+35x^2-3x^2$

$=y^2+12xy+32x^2=y^2+8xy+4xy+32x^2=y(y+8x)+4x(y+8x)$

$=(y+8x)(y+4x)=(x^2+8x+6)(x^2+4x+6)$

2.(vi) $(\mathbf{x+1})(\mathbf{x-1})(\mathbf{x+2})(\mathbf{x-2})+\mathbf{13x^2}$ **wrong statement**

$(\mathbf{x+1})(\mathbf{x-1})(\mathbf{x+2})(\mathbf{x-2})+\mathbf{5x^2}$ **right statement**

$=(x+1)(x+2)(x-1)(x-2)+5x^2$

$=(x^2+3x+2)(x^2-3x+2)+5x^2$

$=(x^2+2+3x)(x^2+2-3x)+5x^2$

$=(y+3x)(y-3x)+5x^2=y^2-9x^2+5x^2=y^2-4x^2$

$=(y-2x)(y+2x)=(x^2-2x+2)(x^2+2x+2)$

3. Factorize:

(i) $8x^3 + 12x^2 + 6x + 1$ (ii) $27a^3 + 108a^2b + 144ab^2 + 64b^3$

(iii) $x^3 + 48x^2y + 108xy^2 + 216y^3$ (iv) $8x^3 - 125y^3 + 150xy^2 - 60x^2y$

Solution

Remember!

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

3.(i) $\boldsymbol{8x^3 + 12x^2 + 6x + 1}$

$= (2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3$

$= (2x+1)^3$

3.(ii) $\boldsymbol{27a^3 + 108a^2b + 144ab^2 + 64b^3}$

$= (3a)^3 + 3(3a)^2(4b) + 3(3a)(4b)^2 + (4b)^3$

$= (3a+4b)^3$

3.(iii) $\boldsymbol{x^3 + 18x^2y + 108xy^2 + 216y^3}$ **wrong statement, i.e. use 18 instead 48**

$= (x)^3 + 3(x)^2(6y) + 3(x)(6y)^2 + (6y)^3$

$= (x+6y)^3$

3.(iv) $\boldsymbol{8x^3 - 125y^3 + 150xy^2 - 60x^2y}$

$= (2x)^3 + (-5y)^3 + 3(2x)(-5y)^2 + 3(2x)^2(-5y)$

$= (2x-5y)^3$

4. Factorize:

(i) $125a^3 - 1$ (ii) $64x^3 + 125$ (iii) $x^6 - 27$

(iv) $1000a^3 + 1$ (v) $343x^3 + 216$ (vi) $27 - 512y^3$

Solution

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

4.(i) $\boldsymbol{125a^3 - 1}$

$= (5a)^3 - (1)^3$

$= (5a-1)[(5a)^2 + (5a)(1) + (1)^2]$

$= (5a-1)(25a^2 + 5a + 1)$

4.(ii) $\boldsymbol{64x^3 + 125}$

$= (4x)^3 + (5)^3$

$= (4x + 5)[(4x)^2 - (4x)(5) + (5)^2]$

$= (4x + 5)(16x^2 - 20x + 25)$

4.(iii) $\boldsymbol{x^6 - 27}$

$= (x^2)^3 - (3)^3$

$= (x^2 - 3)[(x^2)^2 + (x^2)(3) + (3)^2]$

$= (x^2 - 3)(x^4 + 3x^2 + 9)$

4.(iv) $\boldsymbol{1000a^3 + 1}$

$= (10a)^3 + (1)^3$

$= (10a + 1)[(10a)^2 - (10a)(1) + (1)^2]$

$= (10a + 1)(100a^2 - 10a + 1)$

4.(v) $\boldsymbol{343x^3 + 216}$

$= (7x)^3 + (6)^3$

$= (7x + 6)[(7x)^2 - (7x)(6) + (6)^2]$

$= (7x + 6)(49x^2 - 42x + 36)$

4.(vi) $\boldsymbol{27 - 512y^3}$

$= (3)^3 - (8y)^3$

$= (3 - 8y)[(3)^2 + (3)(8y) + (8y)^2]$

$= (3 - 8y)(9 + 24y + 64y^2)$

EXERCISE 4.3

1. Find HCF by factorization method.

(i) $21x^2y, 35xy^2$
(ii) $4x^2-9y^2, 2x^2-3xy$
(iii) x^3-1, x^2+x+1
(iv) $a^3+2a^2-3a, 2a^3+5a^2-3a$
(v) $t^2+3t-4, t^2+5t+4, t^2-1$
(vi) $x^2+15x+56, x^2+5x-24, x^2+8x$

Solution

1.(i) $21x^2y, 35xy^2$

$21x^2y = 3 \times 7 \times x \times x \times y$

$35xy^2 = 5 \times 7 \times x \times y \times y$

$\text{HCF} = 7 \times x \times y = 7xy$

1.(ii) $4x^2-9y^2, 2x^2-3xy$

$4x^2-9y^2 = (2x)^2-(3y)^2 = (2x-3y)(2x+3y)$

$2x^2-3xy = x(2x-3y)$

$\text{HCF} = 2x-3y$

1.(iii) x^3-1, x^2+x+1

$x^3-1 = (x-1)(x^2+x+1)$

$x^2+x+1 = x^2+x+1$

$\text{HCF} = x^2+x+1$

1.(iv) $a^3+2a^2-3a, 2a^3+5a^2-3a$

$a^3+2a^2-3a = a(a^2+2a-3) = a(a^2+3a-a-3) = a(a+3)(a-1)$

$2a^3+5a^2-3a = a(2a^2+5a-3) = a(2a^2+6a-a-3) = a(a+3)(2a-1)$

$\text{HCF} = a(a+3)$

1.(v) $t^2 - 3t - 4, t^2 + 5t + 4, t^2 - 1$ wrong statement in book

$\mathrm{t}^2 - 3\mathrm{t} - 4 = \mathrm{t}^2 - 4\mathrm{t} + \mathrm{t} - 4 = \mathrm{t}(\mathrm{t} - 4) + 1(\mathrm{t} - 4) = (\mathrm{t} - 4)(\mathrm{t} + 1)$

$\mathrm{t}^2 + 5\mathrm{t} + 4 = \mathrm{t}^2 + 4\mathrm{t} + \mathrm{t} + 4 = \mathrm{t}(\mathrm{t} + 4) + 1(\mathrm{t} + 4) = (\mathrm{t} + 4)(\mathrm{t} + 1)$

$\mathrm{t}^2 - 1 = (\mathrm{t} - 1)(\mathrm{t} + 1)$

$\mathrm{HCF} = \mathrm{t} + 1$

1.(vi) $x^2 + 15x + 56, x^2 + 5x - 24, x^2 + 8x$

$\mathrm{x}^2 + 15\mathrm{x} + 56 = \mathrm{x}^2 + 8\mathrm{x} + 7\mathrm{x} + 56 = \mathrm{x}(\mathrm{x} + 8) + 7(\mathrm{x} + 8) = (\mathrm{x} + 8)(\mathrm{x} + 7)$

$\mathrm{x}^2 + 5\mathrm{x} - 24 = \mathrm{x}^2 + 8\mathrm{x} - 3\mathrm{x} - 24 = \mathrm{x}(\mathrm{x} + 8) - 3(\mathrm{x} + 8) = (\mathrm{x} + 8)(\mathrm{x} - 3)$

$\mathrm{x}^2 + 8\mathrm{x} = \mathrm{x}(\mathrm{x} + 8)$

$\mathrm{HCF} = \mathrm{x} + 8$

2. Find HCF of the following expressions by using division method:

(i) $27x^3 + 9x^2 - 3x - 9,\ 3x - 2$ (ii) $x^3 - 9x^2 + 21x - 15,\ x^2 - 4x + 3$

(iii) $2x^3 + 2x^2 + 2x + 2,\ 6x^3 + 12x^2 + 6x + 12$

(iv) $2x^3 - 4x^2 + 6x,\ x^3 - 2x,\ 3x^2 - 6x$

Solution

2.(i) $27x^3 + 9x^2 - 3x - 9, 3x - 2$

Replace with 9

Divisor: $3x - 2$ Quotient: $9x^2 + 9x + 5$

$$27x^3 + 9x^2 - 3x - 10$$

$$\pm 27x^3 \mp 18x^2$$

$$27x^2 - 3x - 10$$

$$\pm 27x^2 \mp 18x$$

$$15x - 10$$

$$\pm 15x \mp 10$$

$$0$$

$\mathrm{HCF} = 3\mathrm{x} - 2$

2.(ii) $\boldsymbol{x^3 - 9x^2 + 21x - 15, x^2 - 4x + 3}$

Replace by 15

$$
\begin{array}{l|l|l}
 & x^3 - 9x^2 + 21x - 9\bullet & \\
x^2 - 4x + 3 & \pm x^3 \mp 4x^2 \pm 3x & x - 5 \\
\hline
 & -5x^2 + 18x - 9 & \\
 & \mp 5x^2 \pm 20x \mp 15 & \\
\hline
 & -2x + 6 &
\end{array}
$$

we may write it $-2(x-3)$

Now

$$
\begin{array}{l|l|l}
 & x^2 - 4x + 3 & \\
x - 3 & \pm x^2 \mp 3x & x - 1 \\
\hline
 & -x + 3 & \\
 & \mp x \pm 3 & \\
\hline
 & 0 &
\end{array}
$$

HCF $=$ x $-$ 3

2.(iii) $\boldsymbol{2x^3 + 2x^2 + 2x + 2, 6x^3 + 12x^2 + 6x + 12}$

or $\quad \boldsymbol{2(x^3 + x^2 + x + 1), 6(x^3 + 2x^2 + x + 2)}$

$$
\begin{array}{l|l|l}
 & x^3 + 2x^2 + x + 2 & \\
x^3 + x^2 + x + 1 & \pm x^3 \pm x^2 \pm x \pm 1 & 1 \\
\hline
 & x^2 + 1 &
\end{array}
$$

Now

$$
\begin{array}{l|l|l}
 & x^3 + x^2 + x + 1 & \\
x^2 + 1 & \pm x^3 \quad \pm x & x + 1 \\
\hline
 & x^2 + 1 & \\
 & \pm x^2 \pm 1 & \\
\hline
 & 0 &
\end{array}
$$

HCF $= 2(x^2 + 1)$ **wrong answer in book**

2.(iv) $\boldsymbol{2x^3 - 4x^2 + 6x, x^3 - 2x, 3x^2 - 6x}$

$$
\begin{array}{r|l|l}
 & 2x^3 - 4x^2 + 6x & \\
x^3 - 2x & \pm 2x^3 \quad \mp 4x & 2 \\
\hline
 & -4x^2 + 10x &
\end{array}
$$

we may write it $-2(2x^2 - 5x)$

Now multiply $x^3 - 2x$ with 2 and simplify

$$
\begin{array}{r|l|l}
 & 2x^3 - 4x & \\
2x^2 - 5x & \pm 2x^3 \mp 5x^2 & x \\
\hline
 & 5x^2 - 4x &
\end{array}
$$

Now multiply $5x^2 - 4x$ with 2 and simplify

$$
\begin{array}{r|l|l}
 & 10x^2 - 8x & \\
2x^2 - 5x & \pm 10x^2 \mp 25x & 5 \\
\hline
 & 17x &
\end{array}
$$

Now

$$
\begin{array}{r|l|l}
 & 2x^2 - 5x & \\
x & \pm 2x^2 \mp 5x & 2x - 5 \\
\hline
 & 0 &
\end{array}
$$

And

$$
\begin{array}{r|l|l}
 & 3x^2 - 6x & \\
x & \pm 3x^2 \mp 6x & 3x - 6 \\
\hline
 & 0 &
\end{array}
$$

$\mathrm{HCF} = \mathrm{x}$ **wrong answer in book**

3. Find LCM of the following expressions by using prime factorization method.

(i) $2a^2b,\ 4ab^2,\ 6ab$ (ii) $x^2+x,\ x^3+x^2$

(iii) $a^2-4a+4,\ a^2-2a$ (iv) $x^4-16,\ x^3-4x$

(v) $16-4x^2,\ x^2+x-6,\ 4-x^2$

Solution

3.(i) $2a^2b, 4ab^2, 6ab$

$2a^2b = 2 \times a \times a \times b$

$4ab^2 = 2 \times 2 \times a \times b \times b$

$6ab = 2 \times 3 \times a \times b$

$\text{Common Factors} = 2 \times a \times b = 2ab$

$\text{non} - \text{ Common Factors} = 2 \times 3 \times a \times b = 6ab$

$\text{LCM} = \text{CF} \times \text{NCF} = 2ab \times 6ab = 12a^2b^2$

3.(ii) $x^2 + x, x^3 + x^2$

$x^2 + x = x(x+1)$

$x^3 + x^2 = x^2(x+1) = x \times x \times (x+1)$

$\text{Common Factors} = x(x+1)$

$\text{non} - \text{ Common Factors} = x$

$\text{LCM} = \text{CF} \times \text{NCF} = x(x+1) \times x = x^2(x+1)$

3.(iii) $a^2 - 4a + 4, a^2 - 2a$

$a^2 - 4a + 4 = (a-2)^2 = (a-2)(a-2)$

$a^2 - 2a = a(a-2)$

$\text{Common Factors} = (a-2)$

$\text{non} - \text{ Common Factors} = a(a-2)$

$\text{LCM} = \text{CF} \times \text{NCF} = (a-2) \times a(a-2) = a(a-2)^2$

3.(iv) $x^4 - 16, x^3 - 4x$

$x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$

$x^3 - 4x = x(x^2 - 4) = x(x - 2)(x + 2)$

$\text{Common Factors} = (x - 2)(x + 2)$

$\text{non} - \text{ Common Factors} = x(x^2 + 4)$

$\text{LCM} = \text{CF} \times \text{NCF} = (x - 2)(x + 2) \times x(x^2 + 4) = x(x^4 - 16)$

3.(v) $16 - 4x^2, x^2 + x - 6, 4 - x^2$

$16 - 4x^2 = 4(4 - x^2) = 4(2 - x)(2 + x)$

$x^2 + x - 6 = x^2 + 3x - 2x - 6 = (x + 3)(x - 2) = -(x + 3)(2 - x)$

$4 - x^2 = (2 - x)(2 + x)$

$\text{Common Factors} = (2 - x)(2 + x)$

$\text{non} - \text{ Common Factors} = -4(x + 3)$

$\text{LCM} = \text{CF} \times \text{NCF} = (2 - x)(2 + x) \times -4(x + 3) = 4(x^2 - 4)(x + 3)$

4. The HCF of two polynomials is $y - 7$ and their LCM is $y^3 - 10y^2 + 11y + 70$. If one of the polynomials is $y^2 - 5y - 14$, find the other.

Solution

$\text{HCF} = y - 7$

$\text{LCM} = y^3 - 10y^2 + 11y + 70$

$p(y) = y^2 - 5y - 14$

$q(y) = ?$

Using formula: $p(y) \times q(y) = \text{HCF} \times \text{LCM}$

$(y^2 - 5y - 14) \times q(y) = (y - 7) \times (y^3 - 10y^2 + 11y + 70)$

$q(y) = \frac{(y-7)\times(y^3-10y^2+11y+70)}{(y^2-5y-14)}$

$q(y) = \frac{(y-7)\times(y^3-10y^2+11y+70)}{(y-7)(y+2)}$

$q(y) = \frac{y^3-10y^2+11y+70}{y+2}$

$$
\begin{array}{r|l|l}
y+2 & y^3-10y^2+11y+70 & y^2-12y+35 \\
 & \pm y^3 \pm 2y^2 & \\
\hline
 & -12y^2+11y+70 & \\
 & \mp 12y^2 \mp 24y & \\
\hline
 & 35y+70 & \\
 & \pm 35y \pm 70 & \\
\hline
 & 0 &
\end{array}
$$

$$\mathbf{q(y)} = \frac{\mathbf{y^3-10y^2+11y+70}}{\boldsymbol{y+2}} = \mathbf{y^2 - 12y + 35}$$

5. The LCM and HCF of two polynomial $p(x)$ and $q(x)$ are $36x^3(x+a)(x^3-a^3)$ and $x^2(x-a)$ respectively. If $p(x) = 4x^2(x^2-a^2)$, find $q(x)$.

Solution

$\text{HCF} = x^2(x-a)$

$\text{LCM} = 36x^3(x+a)(x^3-a^3)$

$p(x) = 4x^2(x^2-a^2)$

$q(x) = ?$

Using formula: $p(x) \times q(x) = \text{HCF} \times \text{LCM}$

$4x^2(x^2-a^2) \times q(x) = x^2(x-a) \times 36x^3(x+a)(x^3-a^3)$

$q(x) = \frac{x^2(x-a)\times 36x^3(x+a)(x^3-a^3)}{4x^2(x^2-a^2)}$

$q(x) = 9x^3(x^3-a^3)$

6. The HCF and LCM of two polynomials is $(x+a)$ and $12x^2(x+a)(x^2-a^2)$ respectively. Find the product of the two polynomials.

Solution

$\text{HCF} = (x+a)$

$\text{LCM} = 12x^2(x+a)(x^2-a^2)$

$p(x) \times q(x) = ?$

Using formula: $p(x) \times q(x) = \text{HCF} \times \text{LCM}$

$p(x) \times q(x) = (x+a) \times 12x^2(x+a)(x^2-a^2)$

$p(x) \times q(x) = 12x^2(x+a)^3(x-a)$ **wrong answer in book**

EXERCISE 4.4

1. Find the square root of the following polynomials by factorization method:

(i) $x^2 - 8x + 16$ (ii) $9x^2 + 12x + 4$

(iii) $36a^2 + 84a + 49$ (iv) $64y^2 - 32y + 4$

(v) $200t^2 - 120t + 18$ (vi) $40x^2 + 120x + 90$

Solution

1.(i) $\sqrt{\boldsymbol{x^2 - 8x + 16}}$ =???

$$x^2 - 8x + 16 = (x)^2 - 2(x)(4) + (4)^2 = (x - 4)^2$$

$$\sqrt{x^2 - 8x + 16} = \sqrt{(x - 4)^2}$$

$$\sqrt{x^2 - 8x + 16} = \pm(x - 4)$$

1.(ii) $\sqrt{\boldsymbol{9x^2 + 12x + 4}}$ =???

$$9x^2 + 12x + 4 = (3x)^2 + 2(3x)(2) + (2)^2 = (3x + 2)^2$$

$$\sqrt{9x^2 + 12x + 4} = \sqrt{(3x + 2)^2}$$

$$\sqrt{9x^2 + 12x + 4} = \pm(3x + 2)$$

1.(iii) $\sqrt{\boldsymbol{36a^2 + 84a + 49}}$ =???

$$36a^2 + 84a + 49 = (6a)^2 + 2(6a)(7) + (7)^2 = (6a + 7)^2$$

$$\sqrt{36a^2 + 84a + 49} = \sqrt{(6a + 7)^2}$$

$$\sqrt{36a^2 + 84a + 49} = \pm(6a + 7)$$

1.(iv) $\sqrt{\boldsymbol{64y^2 - 32y + 4}}$ =???

$$64y^2 - 32y + 4 = (8y)^2 - 2(8y)(2) + (2)^2 = (8y - 2)^2$$

$$\sqrt{64y^2 - 32y + 4} = \sqrt{(8y - 2)^2}$$

$$\sqrt{64y^2 - 32y + 4} = \pm(8y - 2)$$

1.(v) $\sqrt{\mathbf{200}\boldsymbol{t}^{\mathbf{2}} - \mathbf{120}\boldsymbol{t} + \mathbf{18}} = ???$

$200t^2 - 120t + 18 = 2[100t^2 - 60t + 9]$

$200t^2 - 120t + 18 = 2[(10t)^2 - 2(10t)(3) + (3)^2] = 2(10t - 3)^2$

$\sqrt{64y^2 - 32y + 18} = \sqrt{2(10t - 3)^2}$

$\sqrt{64y^2 - 32y + 18} = \pm\sqrt{2}(10t - 3)$

1.(vi) $\sqrt{\mathbf{40}\boldsymbol{x}^{\mathbf{2}} + \mathbf{120}\boldsymbol{x} + \mathbf{90}} = ???$

$40x^2 + 120x + 90 = 10(4x^2 + 12x + 9) = 10[(2x)^2 + 2(2x)(3) + (3)^2]$

$40x^2 + 120x + 90 = 10(2x + 3)^2$

$\sqrt{40x^2 + 120x + 90} = \sqrt{10(2x + 3)^2}$

$\sqrt{40x^2 + 120x + 90} = \pm\sqrt{10}(2x + 3)$

2. Find the square root of the following polynomials by division method:

(i) $4x^4 - 28x^3 + 37x^2 + 42x + 9$

(ii) $121x^4 - 198x^3 - 183x^2 + 216x + 144$

(iii) $x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4$

(iv) $4x^4 - 12x^3 + 37x^2 - 42x + 49$

Solution

1.(i) $\sqrt{\mathbf{4}\boldsymbol{x}^{\mathbf{4}} - \mathbf{28}\boldsymbol{x}^{\mathbf{3}} + \mathbf{37}\boldsymbol{x}^{\mathbf{2}} + \mathbf{42}\boldsymbol{x} + \mathbf{9}} = ???$

	$2x^2 - 7x - 3$
	$4x^4 - 28x^3 + 37x^2 + 42x + 9$
$2x^2$	$\pm 4x^4$
	$-28x^3 + 37x^2$
$4x^2 - 7x$	$\mp 28x^3 \pm 49x^2$
	$-12x^2 + 42x + 9$
$4x^2 - 14x - 3$	$\mp 12x^2 \pm 42x \pm 9$
	0

$\sqrt{4x^4 - 28x^3 + 37x^2 + 42x + 9} = \pm(2x^2 - 7x - 3)$

1.(ii) $\sqrt{\boldsymbol{121x^4 - 198x^3 - 183x^2 + 216x + 144}} = ???$

	$11x^2 - 9x - 12$
$11x^2$	$121x^4 - 198x^3 - 183x^2 + 216x + 144$ $\pm 121x^4$
$22x^2 - 9x$	$-198x^3 - 183x^2$ $\mp 198x^3 \pm 81x^2$
$22x^2 - 18x - 12$	$-264x^2 + 216x + 144$ $\mp 264x^2 \pm 216x \pm 144$
	0

$$\sqrt{121x^4 - 198x^3 - 183x^2 + 216x + 144} = \pm(11x^2 - 9x - 12)$$

1.(iii) $\sqrt{\boldsymbol{x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4}} = ???$

	$x^2 - 5xy + y^2$
x^2	$x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4$ $\pm x^4$
$2x^2 - 5xy$	$-10x^3y + 27x^2y^2$ $\mp 10x^3y \pm 25x^2y^2$
$2x^2 - 10xy + y^2$	$2x^2y^2 - 10xy^3 + y^4$ $\pm 2x^2y^2 \mp 10xy^3 \pm y^4$
	0

$$\sqrt{x^4 - 10x^3y + 27x^2y^2 - 10xy^3 + y^4} = \pm(x^2 - 5xy + y^2)$$

1.(iv) $\sqrt{\boldsymbol{4x^4 - 12x^3 + 37x^2 - 42x + 49}} = ???$

	$2x^2 - 3x + 7$
$2x^2$	$4x^4 - 12x^3 + 37x^2 - 42x + 49$ $\pm 4x^4$
$4x^2 - 3x$	$-12x^3 + 37x^2$ $\mp 12x^3 \pm 9x^2$
$4x^2 - 6x + 7$	$28x^2 - 42x + 49$ $\pm 28x^2 \mp 42x \pm 49$
	0

$$\sqrt{4x^4 - 12x^3 + 37x^2 - 42x + 49} = \pm(2x^2 - 3x + 7)$$

3. An investor's return $R(x)$ in rupees after investing x thousand rupees is given by quadratic expression:

$$R(x) = -x^2 + 6x - 8$$

Factorize the expression and find the investment levels that result in zero return.

Solution

$R(x) = -x^2 + 6x - 8 = -x^2 + 4x + 2x - 8 = -x(x - 4) + 2(x - 4)$

$R(x) = (-x + 2)(x - 4)$

For zero return $R(x) = 0$ we have $(-x + 2)(x - 4) = 0$

$-x + 2 = 0$ or $x - 4 = 0$

$x = 2$ or $x = 4$

Investment levels that result in zero return will be $x = 2$ and $x = 4$

4. A company's profit $P(x)$ in rupees from selling x units of a product is modeled by the cubic expression:

$$P(x) = x^3 - 15x^2 + 75x - 125$$

Find the break-even point(s), where the profit is zero.

Solution

$P(x) = x^3 - 15x^2 + 75x - 125$

$P(x) = (x)^3 - 3(x)^2(5) + 3(x)(5)^2 - (5)^3 = (x - 5)^3$

Since profit is zero, using $P(x) = 0$ we have $(x - 5)^3 = 0$

After taking cube root on both sides we have $x = 5$

5. The potential energy $V(x)$ in an electric field varies as a cubic function of distance x, given by:

$$V(x) = 2x^3 - 6x^2 + 4x$$

Determine where the potential energy is zero.

Solution

$V(x) = 2x^3 - 6x^2 + 4x$

$V(x) = 2x(x^2 - 3x + 2) = 2x(x - 2)(x - 1)$

For zero potential energy, using $V(x) = 0$ we have $2x(x - 1)(x - 2) = 0$

Then $x = 0, x = 1, x = 2$

6. In structural engineering, the deflection $Y(x)$ of a beam is given by:

$$Y(x) = 2x^2 - 8x + 6$$

This equation gives the vertical deflection at any point x along the beam. Find the points of zero deflection.

Solution

$$Y(x) = 2x^2 - 8x^2 + 6$$

$$Y(x) = 2(x^2 - 4x + 3)$$

$$Y(x) = 2(x^2 - 3x - x + 3)$$

$$Y(x) = 2(x - 1)(x - 3)$$

For zero potential deflection, using $Y(x) = 0$ we have $2(x - 1)(x - 3) = 0$

$2 \neq 0$ then $x - 1 = 0$ or $x - 3 = 0$

Then $x = 1, x = 3$

REVIEW EXERCISE 4

1. Four options are given against each statement. Encircle the correct option.

   i. The factorization of $12x + 36$ is:

   (a) ✓ $12(x + 3)$ (b) $12(3x)$ (c) $12(3x + 1)$ (d) $x(12 + 36x)$

   ii. The factors of $4x^2 - 12y + 9$ are:

   (a) $(2x + 3)^2$ (b) ✓ $(2x - 3)^2$
   (c) $(2x - 3)(2x + 3)$ (d) $(2 + 3x)(2 - 3x)^2$

   iii. The HCF of a^3b^3 and ab^2 is:

   (a) a^3b^3 (b) ✓ ab^2 (c) a^4b^5 (d) a^2b

   iv. The LCM of $16x^2$, $4x$ and $30xy$ is:

   (a) $480x^3y$ (b) $240xy$ (c) ✓ $240x^2y$ (d) $120x^4y$

   v. Product of LCM and HCF = __________ of two polynomials.

   (a) sum (b) difference (c) ✓ product (d) quotient

   vi. The square root of $x^2 - 6x + 9$ is:

   (a) ✓ $\pm(x - 3)$ (b) $\pm(x + 3)$ (c) $x - 3$ (d) $x + 3$

   vii. The LCM of $(a - b)^2$ and $(a - b)^4$ is:

   (a) $(a - b)^2$ (b) $(a - b)^3$ (c) ✓ $(a - b)^4$ (d) $(a - b)^6$

   viii. Factorization of $x^3 + 3x^2 + 3x + 1$ is:

   (a) ✓ $(x + 1)^3$ (b) $(x - 1)^3$
   (c) $(x + 1)(x^2 + x + 1)$ (d) $(x - 1)(x^2 - x + 1)$

   ix. Cubic polynomial has degree:

   (a) 1 (b) 2 (c) ✓ 3 (d) 4

   x. One of the factors of $x^3 - 27$ is:

   (a) ✓ $x - 3$ (b) $x + 3$ (c) $x^2 - 3x + 9$ (d) Both *a* and *c*

2. Factorize the following expressions:

(i) $4x^3 + 18x^2 - 12x$ (ii) $x^3 + 64y^3$

(iii) $x^3y^3 - 8$ (iv) $-x^2 - 23x - 60$

(v) $2x^2 + 7x + 3$ (vi) $x^4 + 64$

(vii) $x^4 + 2x^2 + 9$ (viii) $(x + 3)(x + 4)(x + 5)(x + 6) - 360$

(ix) $(x^2 + 6x + 3)(x^2 + 6x - 9) + 36$

Solution

2.(i) $4x^3 + 18x^2 - 12x$

$= 2x(2x^2 + 9x - 6)$

2.(ii) $x^3 + 64y^3$

$= (x)^3 + (4y)^3$

$= (x + 4y)[(x)^2 - (x)(4y) + (4y)^2]$

$= (x + 4y)(x^2 - 4xy + 16y^2)$

2.(iii) $x^3y^3 - 8$

$= (xy)^3 - (2)^3$

$= (xy - 2)[(xy)^2 + (xy)(2) + (2)^2]$

$= (xy - 2)(x^2y^2 + 2xy + 4)$

2.(iv) $-x^2 - 23x - 60$

$= -(x^2 + 23x + 60) = -[x^2 + 20x + 3x + 60] = -[x(x + 20) + 3(x + 20)]$

$= -(x + 3)(x + 20)$

2.(v) $2x^2 + 7x + 3$

$= 2x^2 + 6x + x + 3 = 2x(x + 3) + 1(x + 3)$

$= (2x + 1)(x + 3)$

2.(vi) $\boldsymbol{x^4 + 64}$

$= (x^2)^2 + (8)^2 + 2(x^2)(8) - 2(x^2)(8) = (x^2 + 8)^2 - 16x^2$

$= (x^2 + 8)^2 - (4x)^2 = (x^2 + 8 - 4x)(x^2 + 8 + 4x)$

$= (x^2 - 4x + 8)(x^2 + 4x + 8)$

2.(vii) $\boldsymbol{x^4 + 2x^2 + 9}$

$= (x^2)^2 + 2x^2 + (3)^2$

$= (x^2)^2 + 2(x^2)(3) + (3)^2 + 2x^2 - 2(x^2)(3)$

$= (x^2 + 3)^2 - 4x^2$

$= (x^2 + 3)^2 - (2x)^2 = (x^2 + 3 - 2x)(x^2 + 3 + 2x)$

$= (x^2 - 2x + 3)(x^2 + 2x + 3)$

2.(viii) $\boldsymbol{(x + 3)(x + 4)(x + 5)(x + 6) - 360}$

$= (x + 3)(x + 6)(x + 4)(x + 5) - 360$

$= (x^2 + 9x + 18)(x^2 + 9x + 20) - 360$

$= (y + 18)(y + 20) - 360 = y^2 + 38y + 360 - 360 = y^2 + 38y$

$= y(y + 38) = (x^2 + 9x)(x^2 + 9x + 38)$

$= x(x + 9)(x^2 + 9x + 38)$

2.(ix) $\boldsymbol{(x^2 + 6x + 3)(x^2 + 6x - 9) + 36}$

$= (x^2 + 6x + 3)(x^2 + 6x - 9) + 36$

$= (y + 3)(y - 9) + 36 = y^2 - 6y - 27 + 36 = y^2 - 6y + 9$

$= (y)^2 - 2(y)(3) + (3)^2 = (y - 3)^2$

$= (x^2 + 6x - 3)^2$

3. Find LCM and HCF by prime factorization method:

(i) $4x^3 + 12x^2,\ 8x^2 + 16x$ (ii) $x^3 + 3x^2 - 4x,\ x^2 - x - 6$

(iii) $x^2 + 8x + 16,\ x^2 - 16$ (iv) $x^3 - 9x,\ x^2 - 4x + 3$

Solution

3.(i) $4x^3 + 12x^2, 8x^2 + 16x$

$4x^3 + 12x^2 = 4x^2(x+3) = 4 \times x \times x \times (x+3)$

$8x^2 + 16x = 8x(x+2) = 2 \times 4 \times x \times (x+2)$

$\text{Common Factors} = 4x$

$\text{Un} - \text{ Common Factors} = x \times (x+3) \times 2 \times (x+2) = 2x(x+3)(x+2)$

$\text{HCF} = 4x$

$\text{LCM} = \text{CF} \times \text{UCF} = 4x \times 2x(x+3)(x+2) = 8x^2(x+2)(x+3)$

3.(ii) $x^3 + 3x^2 - 4x, x^2 - x - 6$

$x^3 + 3x^2 - 4x = x(x^2 + 3x - 4) = x(x^2 + 4x - x - 4) = x(x-1)(x+4)$

$x^2 - x - 6 = x^2 - 3x + 2x - 6 = (x-3)(x+2)$

$\text{Common Factors} = 1$

$\text{Un} - \text{ Common Factors} = x(x-1)(x+4)(x-3)(x+2)$

$\text{HCF} = 1$

$\text{LCM} = \text{CF} \times \text{UCF} = x(x-1)(x+2)(x-3)(x+4)$ **wrong answer in book**

3.(iii) $x^2 + 8x + 16, x^2 - 16$

$x^2 + 8x + 16 = (x)^2 + 2(x)(4) + (4)^2 = (x+4)^2 = (x+4)(x+4)$

$x^2 - 16 = (x)^2 - (4)^2 = (x-4)(x+4)$

$\text{Common Factors} = (x+4)$

$\text{Un} - \text{ Common Factors} = (x-4)(x+4)$

$\text{HCF} = (x+4)$

$\text{LCM} = \text{CF} \times \text{UCF} = (x+4) \times (x-4)(x+4) = (x-4)(x+4)^2$

3.(iv) $x^3 - 9x, x^2 - 4x + 3$

$x^3 - 9x = x(x^2 - 9) = x(x-3)(x+3)$

$x^2 - 4x + 3 = x^2 - 3x - x + 3 = (x-3)(x-1)$

$\text{Common Factors} = (x-3)$

$\text{Un} - \text{ Common Factors} = x(x+3)(x-1)$

$\text{HCF} = (x-3)$

$\text{LCM} = \text{CF} \times \text{UCF} = (x-3) \times x(x+3)(x-1)$

$\text{LCM} = x(x-1)(x^2-9)$ **wrong answer in book**

4. Find square root by factorization and division method of the expression $16x^4 + 8x^2 + 1$.

Solution

Factorization:

$16x^4 + 8x^2 + 1 = (4x^2)^2 + 2(4x^2)(1) + (1)^2$

$16x^4 + 8x^2 + 1 = (4x^2 + 1)^2$

$\sqrt{16x^4 + 8x^2 + 1} = \sqrt{(4x^2+1)^2}$

$\sqrt{16x^4 + 8x^2 + 1} = \pm(4x^2 + 1)$

Division Method:

	$4x^2 + 1$
	$16x^4 + 8x^2 + 1$
$4x^2$	$16x^4$
	$8x^2 + 1$
$8x^2 + 1$	$\pm 8x^2 \pm 1$
	0

$\sqrt{16x^4 + 8x^2 + 1} = \pm(4x^2 + 1)$

5. Huria is analyzing the total cost of her loan, modeled by the expression $C(x) = x^2 - 8x + 15$, where x represents the number of years. What is the optimal repayment period for Huria's loan?

Solution

$C(x) = x^2 - 8x + 15 = x^2 - 5x - 3x + 15 = (x-5)(x-3)$

For optimal repayment, using $C(x) = 0$ we have $(x-5)(x-3) = 0$

We have $x = 3, x = 5$. That is 3 years or 5 years.

Unit 5 Linear Equations and Inequalities

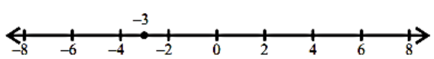
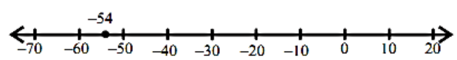
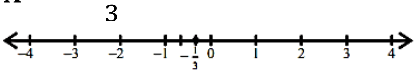
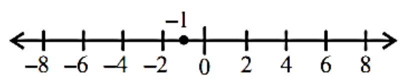
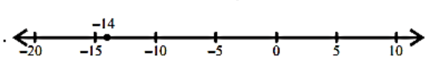
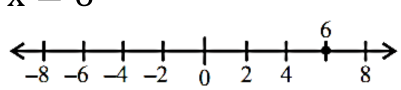
EXERCISE 5.1

1. Solve and represent the solution on a real line.

(i) $12x + 30 = -6$ (ii) $\frac{x}{3} + 6 = -12$ (iii) $\frac{x}{2} - \frac{3x}{4} = \frac{1}{12}$

(iv) $2 = 7(2x + 4) + 12x$ (v) $\frac{2x-1}{3} - \frac{3}{4}\frac{x}{} = \frac{5}{6}$ (vi) $\frac{-5x}{10} = 9 - \frac{10}{5}x$

Solution

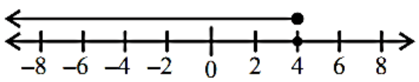
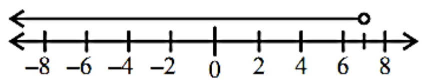
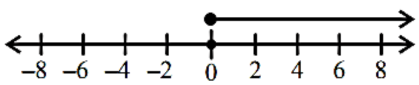
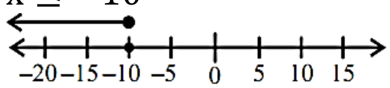
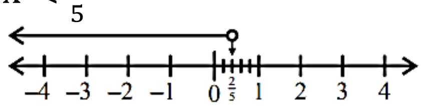
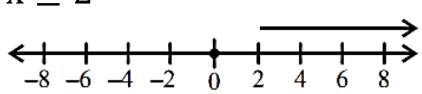
(i) $12x + 30 = -6$ $12x = -6 - 30$ $12x = -36$ $x = -\frac{36}{12}$ $x = -3$  	**(ii)** $\frac{x}{3} + 6 = -12$ $\frac{x}{3} = -12 - 6$ $\frac{x}{3} = -18$ $x = -18 \times 3 \Rightarrow x = -54$ 
(iii) $\frac{x}{2} - \frac{3x}{4} = \frac{1}{12}$ $12 \times \left(\frac{x}{2}\right) - 12 \times \left(\frac{3x}{4}\right) = 12 \times \left(\frac{1}{12}\right)$ $6x - 9x = 1 \Rightarrow -3x = 1$ $x = -\frac{1}{3}$  	**(iv)** $2 = 7(2x + 4) + 12x$ $2 = 14x + 28 + 12x$ $2 - 28 = 14x + 12x$ $-26 = 26x \Rightarrow x = -\frac{26}{26}$ $x = -1$ 
(v) $\frac{2x-1}{3} - \frac{3x}{4} = \frac{5}{6}$ $12 \times \left(\frac{2x-1}{3}\right) - 12 \times \left(\frac{3x}{4}\right) = 12 \times \left(\frac{5}{6}\right)$ $4(2x - 1) - 9x = 10$ $8x - 4 - 9x = 10$ $8x - 9x = 10 + 4 \Rightarrow x = -14$  	**(vi)** $-\frac{5x}{10} = 9 - \frac{10}{5}x$ $10 \times \left(-\frac{5x}{10}\right) = 10 \times (9) - 10 \times \left(\frac{10}{5}x\right)$ $-5x = 90 - 20x$ $-5x + 20x = 90 \Rightarrow 15x = 90$ $x = 6$ 

2. Solve each inequality and represent the solution on a real line.

(i) $x-6\le-2$ (ii) $-9>-16+x$ (iii) $3+2x\ge3$

(iv) $6(x+10)\le0$ (v) $\frac{5}{3}x-\frac{3}{4}<\frac{-1}{12}$ (vi) $\frac{1}{4}x-\frac{1}{2}\le-1+\frac{1}{2}x$

Solution

(i) $x-6\le-2$ $x\le-2+6$ $x\le4$ 	**(ii)** $-9>-16+x$ $-9+16>x$ $7>x$ or $x<7$ 
(iii) $3+2x\ge3$ $2x\ge3-3$ $2x\ge0$ $x\ge0$ 	**(iv)** $6(x+10)\le0$ $6x+60\le0$ $6x\le-60$ $x\le-\frac{60}{6}$ $x\le-10$ 
(v) $\frac{5}{3}x-\frac{3}{4}<-\frac{1}{12}$ $12\times\left(\frac{5}{3}x\right)-12\times\left(\frac{3}{4}\right)<12\times\left(-\frac{1}{12}\right)$ $4(5x)-9<-1$ $20x<-1+9$ $20x<8$ $x<\frac{8}{20}$ $x<\frac{2}{5}$ 	**(vi)** $\frac{1}{4}x-\frac{1}{2}\le-1+\frac{1}{2}x$ $4\times\left(\frac{1}{4}x\right)-4\times\left(\frac{1}{2}\right)\le4\times(-1)+4\times\left(\frac{1}{2}x\right)$ $x-2\le-4+2x$ $-2+4\le2x-x$ $2\le x$ $x\ge2$ 

3. Shade the solution region for the following linear inequalities in xy-plane:

(i)	$2x+y \le 6$	(ii)	$3x+7y \ge 21$	(iii)	$3x-2y \ge 6$
(iv)	$5x-4y \le 20$	(v)	$2x+1 \ge 0$	(vi)	$3y-4 \le 0$

Solution

3 (i) $2x + y \le 6$

Associated equations: $2x + y = 6$

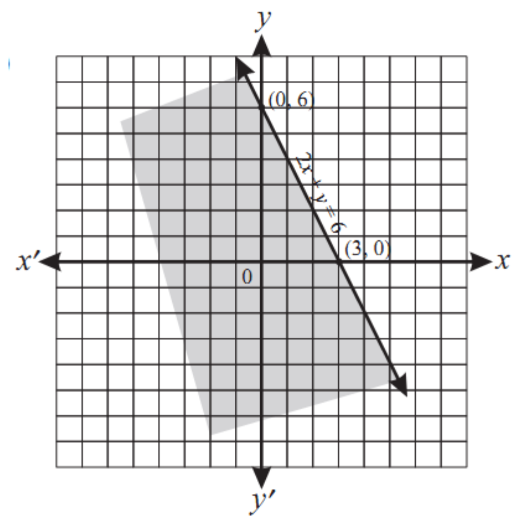
To find Points:

Put $x = 0$, $y = 6$ then point is $(0,6)$

Put $y = 0$, $x = 3$ then point is $(3,0)$

To check Region put $(0,0)$ in given eq.

$0 < 6$ true, graph towards the origin



3 (ii) $3x + 7y \ge 21$

Associated equations: $3x + 7y = 21$

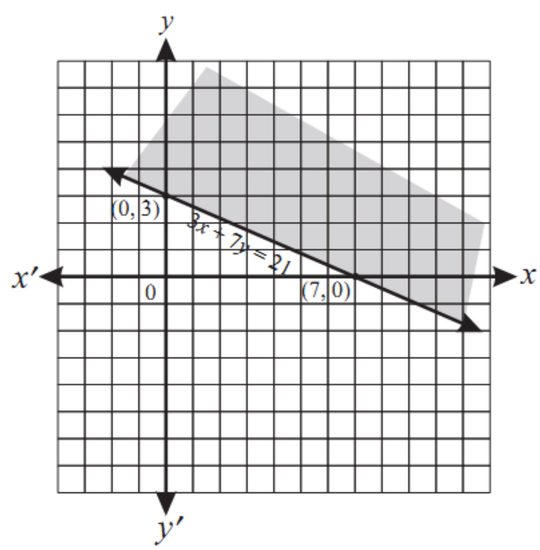
To find Points

Put $x = 0$, $y = 3$ then point is $(0,3)$

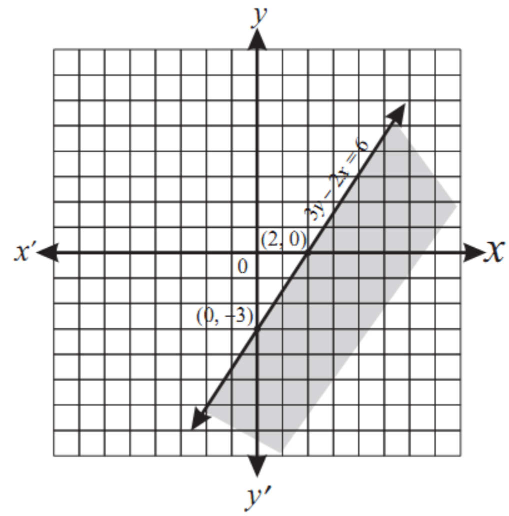
Put $y = 0$, $x = 7$ then point is $(7,0)$

To check Region put $(0,0)$ in given eq.

$0 > 21$ false, graph away from origin



3 (iii) $3x - 2y \geq 6$



Associated equations: $3x - 2y = 6$

To find Points:

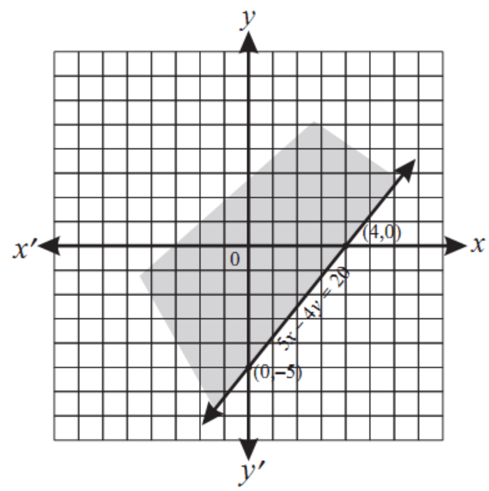
Put $x = 0$, $y = -3$ then point is $(0, -3)$

Put $y = 0$, $x = 2$ then point is $(2,0)$

To check Region put $(0,0)$ in given eq.

$0 > 6$ false, graph away from origin

3 (iv) $5x - 4y \leq 20$



Associated equations: $5x - 4y = 20$

To find Points

Put $x = 0$, $y = -5$ then point is $(0, -5)$

Put $y = 0$, $x = 4$ then point is $(4,0)$

To check Region put $(0,0)$ in given eq.

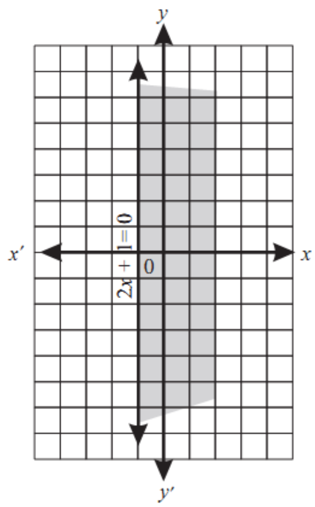
$0 < 20$ true, graph towards the origin

3 (v) $2x + 1 \geq 0$

Associated equations: $2x + 1 = 0$

Point: $x = -\frac{1}{2}$

Region: $1 > 0$ true, graph towards the origin

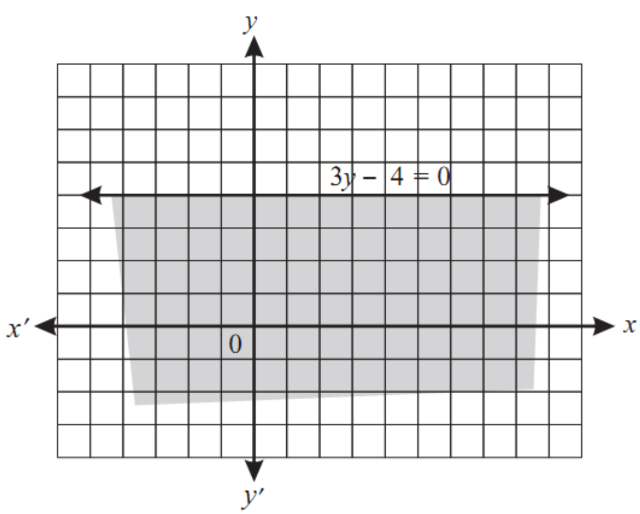


3 (vi) $3y - 4 \leq 0$

Associated equations: $3y - 4 = 0$

Point: $y = \frac{4}{3}$

Region: $0 < 4$ true, graph towards the origin



4. Indicate the solution region of the following linear inequalities by shading:

(i) $2x-3y\le 6$, $2x+3y\le 12$ (ii) $x+y\ge 5$, $-y+x\le 1$ (iii) $3x+7y\ge 21$, $x-y\le 2$

(iv) $4x-3y\le 12$, $x\ge -\frac{3}{2}$ (v) $3x+7y\ge 21$, $y\le 4$ (vi) $5x+7y\le 35$, $x-2y\le 2$

Solution

4 (i)

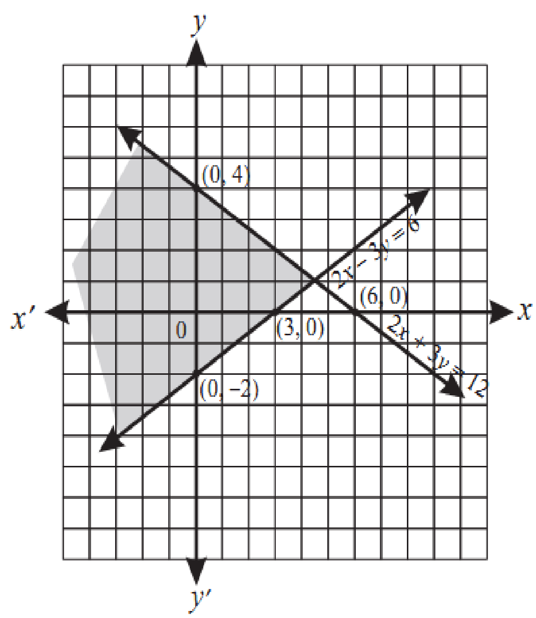
$\mathbf{2x-3y\le 6}$ ………(i)

$\mathbf{2x+3y\le 12}$ ………(ii)

Associated equations

$2x-3y=6$ ………(iii)

$2x+3y=12$ ………(iv)



To find Points

(iii) $\Rightarrow$ Put $x=0$, $y=-2$ then point is $(0,-2)$

(iii) $\Rightarrow$ Put $y=0$, $x=3$ then point is $(3,0)$

(iv) $\Rightarrow$ Put $x=0$, $y=4$ then point is $(0,4)$

(iv) $\Rightarrow$ Put $y=0$, $x=6$ then point is $(6,0)$

To check Region put $(0,0)$ in (i) and (ii)

(i) $\Rightarrow 0<6$ true, graph towards the origin

(ii) $\Rightarrow 0<12$ true, graph towards the origin

4 (ii)

$\mathbf{x + y \geq 5 \quad ....(i)}$

$\mathbf{-y + x \leq 1 \quad ....(ii)}$

Associated equations

$x + y = 5 \quad ....(iii)$

$x - y = 1 \quad ....(iv)$

To find Points

$(iii) \Rightarrow$ Put $x = 0$, $y = 5$ then point is $(0,5)$

$(iii) \Rightarrow$ Put $y = 0$, $x = 5$ then point is $(5,0)$

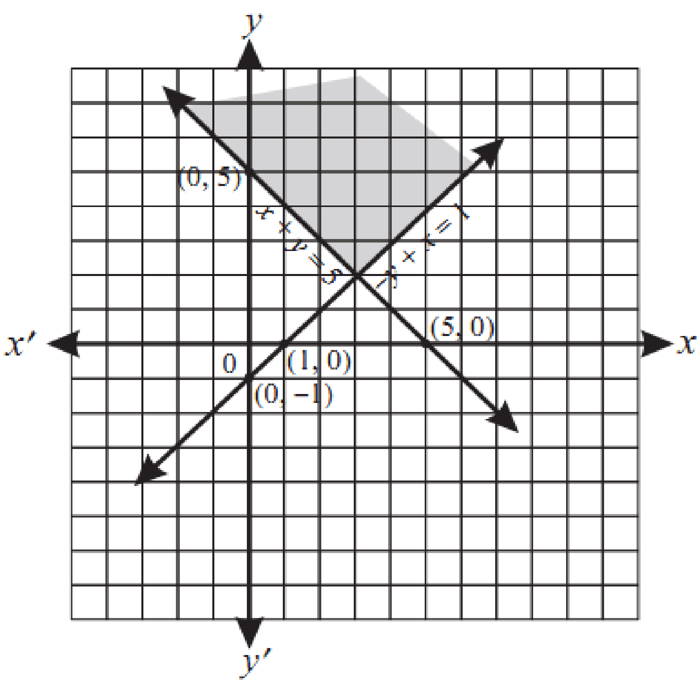
$(iv) \Rightarrow$ Put $x = 0$, $y = -1$ then point is $(0,-1)$

$(iv) \Rightarrow$ Put $y = 0$, $x = 1$ then point is $(1,0)$

To check Region put $(\mathbf{0,0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

$(i) \Rightarrow 0 > 5$ false, graph away from origin

$(ii) \Rightarrow 0 < 1$ true, graph towards the origin



4 (iii)

$\mathbf{3x + 7y \geq 21}$ ………(i)

$\mathbf{x - y \leq 2}$ ………(ii)

Associated equations

$3x + 7y = 21$ ………(iii)

$x - y = 2$ ………(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 3$ then point is $(0,3)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 7$ then point is $(7,0)$

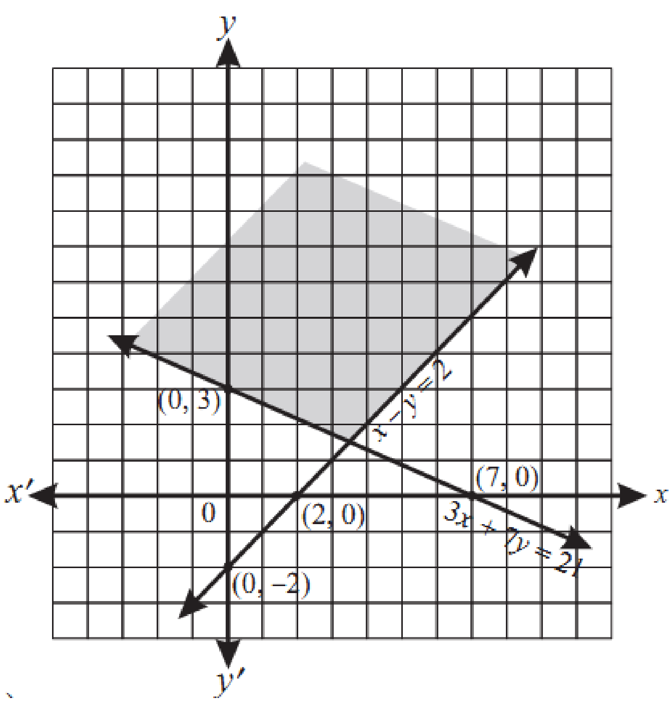
(iv) $\Rightarrow$ Put $x = 0$, $y = -2$ then point is $(0,-2)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 2$ then point is $(2,0)$

To check Region put $(\mathbf{0},\mathbf{0})$ in (i) and (ii)

(i) $\Rightarrow 0 > 21$ false, graph away from origin

(ii) $\Rightarrow 0 < 2$ true, graph towards the origin



4 (iv)

$\mathbf{4x - 3y \leq 12}$ **.........(i)**

$\mathbf{x \geq -\frac{3}{2}}$ **.........(ii)**

Associated equations

$4x - 3y = 12$ ....(iii)

$x = -\frac{3}{2}$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = -4$ then point is $(0, -4)$

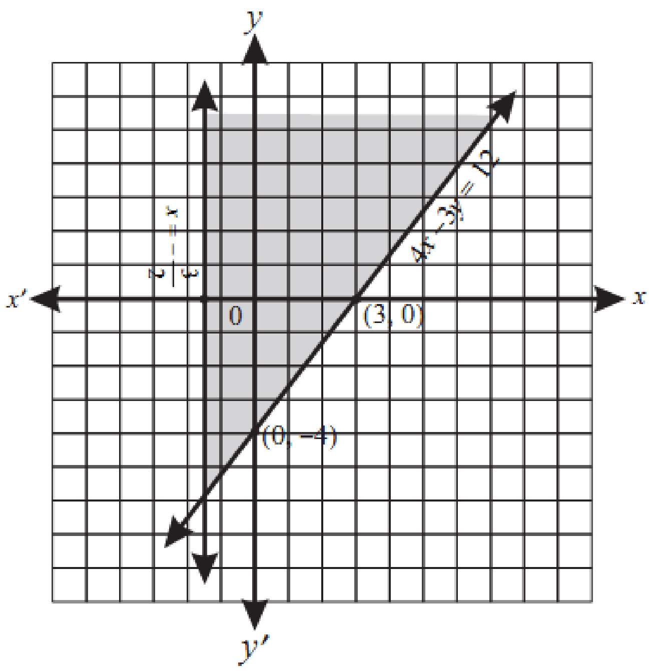
(iii) $\Rightarrow$ Put $y = 0$, $x = 3$ then point is $(3,0)$

(iv) $\Rightarrow$ we have $y = 0$, $x = -\frac{3}{2}$ then point is $\left(-\frac{3}{2}, 0\right)$

To check Region put $(\mathbf{0}, \mathbf{0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) $\Rightarrow 0 < 12$ true, graph towards the origin

(ii) $\Rightarrow 0 > -\frac{3}{2}$ true, graph towards the origin



4 (v)

$\mathbf{3x + 7y \geq 12}$ **.........(i)**

$\mathbf{y \leq 4}$ **.........(ii)**

Associated equations

$3x + 7y = 12$ ....(iii)

$y = 4$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 3$ then point is $(0,3)$

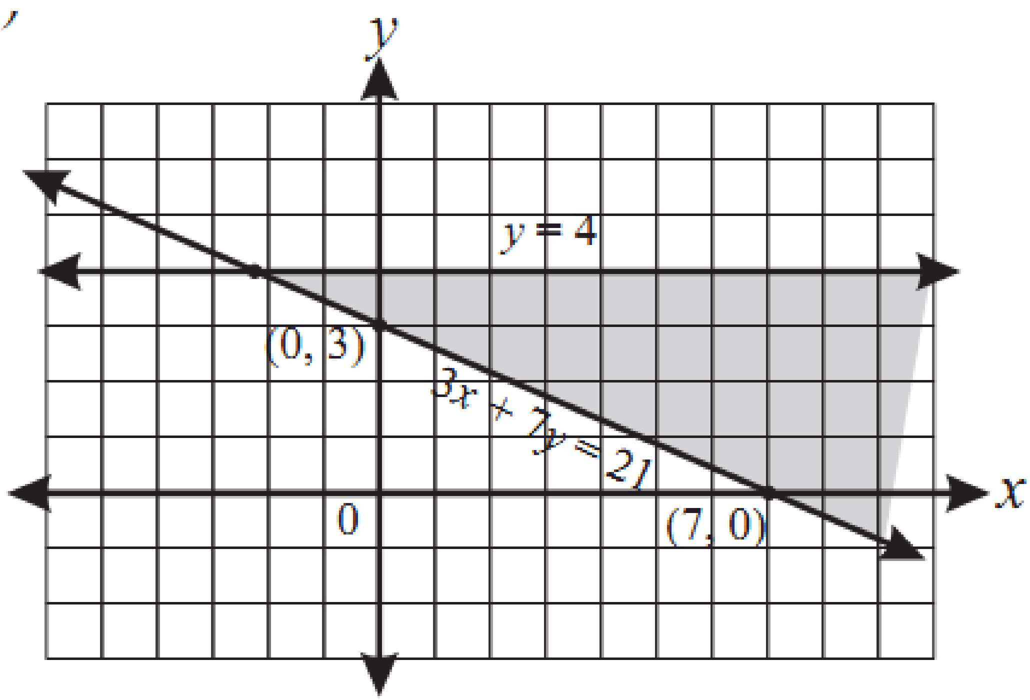
(iii) $\Rightarrow$ Put $y = 0$, $x = 7$ then point is $(7,0)$

(iv) $\Rightarrow$ we have $x = 0$, $y = 4$ then point is $(0,4)$

To check Region put $(\mathbf{0,0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) $\Rightarrow 0 > 12$ false, graph away from origin

(ii) $\Rightarrow 0 < 4$ true, graph towards the origin



4 (vi)

$\mathbf{5x + 7y \leq 35}$ **.........(i)**

$\mathbf{x - 2y \leq 2}$ **.........(ii)**

Associated equations

$5x + 7y = 35$ ....(iii)

$x - 2y = 2$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 5$ then point is $(0,5)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 7$ then point is $(7,0)$

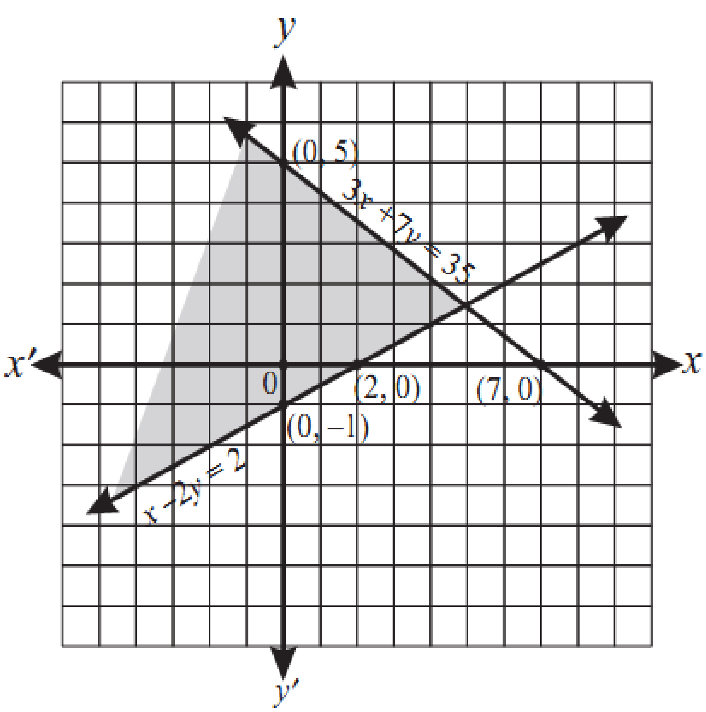
(iv) $\Rightarrow$ Put $x = 0$, $y = -1$ then point is $(0,-1)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 2$ then point is $(2,0)$

To check Region put $(\mathbf{0},\mathbf{0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) $\Rightarrow 0 < 35$ true, graph towards the origin

(ii) $\Rightarrow 0 < 2$ true, graph towards the origin



EXERCISE 5.2

1. Maximize $f(x, y) = 2x + 5y$; subject to the constraints

$$2y - x \le 8 \quad ; \quad x - y \le 4 \quad ; \quad x \ge 0; \ \ y \ge 0$$

Solution

$-x + 2y \le 8$ ………(i)

$x - y \le 4$ ………(ii)

Associated equations

$-x + 2y = 8$ ………(iii)

$x - y = 4$ ………(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 4$ then point is $(0,4)$

(iii) $\Rightarrow$ Put $y = 0$, $x = -8$ then point is $(-8,0)$

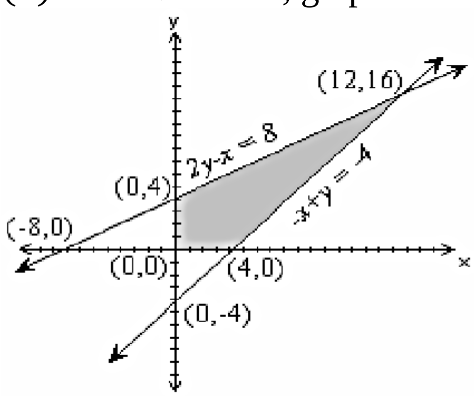
(iv) $\Rightarrow$ Put $x = 0$, $y = -4$ then point is $(0,-4)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 4$ then point is $(4,0)$

To check Region put $(\mathbf{0}, \mathbf{0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) $\Rightarrow 0 < 8$ true, graph towards the origin

(ii) $\Rightarrow 0 < 4$ true, graph towards the origin



Solve $(\mathbf{iii}) + (\mathbf{iv})$

$(-x + 2y) + (x - y) = 8 + 4$ we have $y = 12$

Put $y = 12$ in (iii) we have $x = 16$ and $D(16,12)$

Corner Points of Feasible Region: $A(0,0), B(4,0), C(0,4), D(16,12)$

At A: $z = f(0,0) = 2(0) + 5(0) = 0$

At B: $z = f(4,0) = 2(4) + 5(0) = 8$

At C: $z = f(0,4) = 2(0) + 5(4) = 20$

At D: $z = f(16,12) = 2(16) + 5(12) = 92$

So $z = 2x + 5y$ is maximum at $(16,12)$

2. Maximize $f(x, y) = x + 3y$; subject to the constraints

$$2x + 5y \leq 30 \quad ; \quad 5x + 4y \leq 20 \quad ; \quad x \geq 0 \, ; y \geq 0$$

Solution

$2x + 5y \leq 30$ ………(i)

$5x + 4y \leq 20$ ………(ii)

Associated equations

$2x + 5y = 30$ ………(iii)

$5x + 4y = 20$ ………(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 6$ then point is $(0,6)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 15$ then point is $(15,0)$

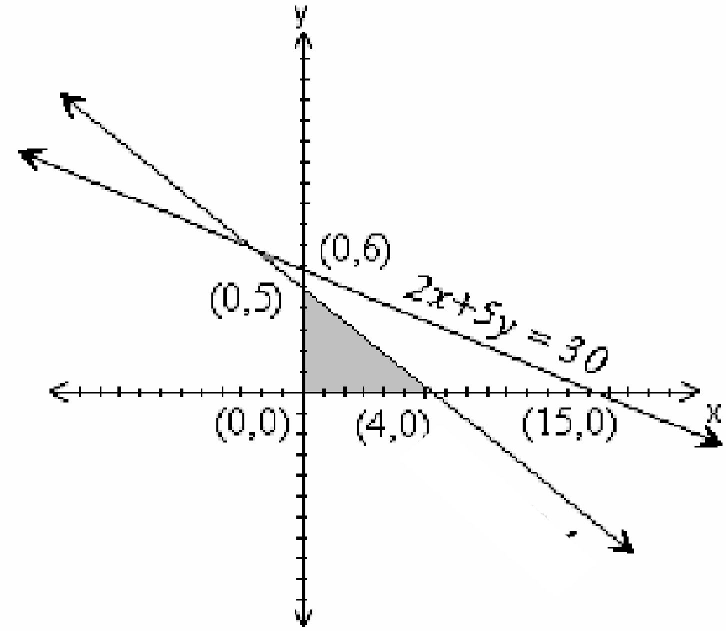
(iv) $\Rightarrow$ Put $x = 0$, $y = 5$ then point is $(0,5)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 4$ then point is $(4,0)$

To check Region put $(\mathbf{0}, \mathbf{0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) $\Rightarrow 0 < 30$ true, graph towards the origin

(ii) $\Rightarrow 0 < 20$ true, graph towards the origin



Corner Points of Feasible Region: $A(0,0), B(4,0), C(0,5)$

At A: $z = f(0,0) = (0) + 3(0) = 0$

At B: $z = f(4,0) = (4) + 3(0) = 4$

At C: $z = f(0,5) = (0) + 3(5) = 15$

So $z = x + 3y$ is maximum at $(0,5)$

3. Maximize $z = 2x + 3y$; subject to the constraints:

$$2x + y \leq 4 \quad ; \quad 4x - y \leq 4 \quad ; \quad x \geq 0: \;\; y \geq 0$$

Solution

$2x + y \leq 4$ ....(i)

$4x - y \leq 4$ ....(ii)

Associated equations

$2x + y = 4$ ....(iii)

$4x - y = 4$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 4$ then point is $(0,4)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 2$ then point is $(2,0)$

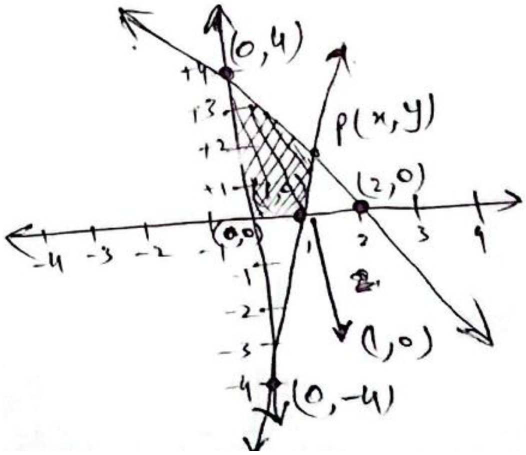
(iv) $\Rightarrow$ Put $x = 0$, $y = -4$ then point is $(0,-4)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 1$ then point is $(1,0)$

To check Region put $(\mathbf{0},\mathbf{0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) $\Rightarrow 0 < 4$ true, graph towards the origin

(ii) $\Rightarrow 0 < 4$ true, graph towards the origin



Solve $(\mathbf{iii}) + (\mathbf{iv})$

$(2x + y) + (4x - y) = 4 + 4$ we have $x = \frac{4}{3}$

Put $x = \frac{4}{3}$ in (iii) we have $y = \frac{4}{3}$ and the intersecting point is $\left(\frac{4}{3}, \frac{4}{3}\right)$

Corner Points: $A(0,0), B(1,0), C(0,4), P\left(\frac{4}{3}, \frac{4}{3}\right)$

At A: $z = f(0,0) = 2(0) + 3(0) = 0$

At B: $z = f(1,0) = 2(1) + 3(0) = 2$

At C: $z = f(0,4) = 2(0) + 3(4) = 12$

At P: $z = f\left(\frac{4}{3}, \frac{4}{3}\right) = 2\left(\frac{4}{3}\right) + 3\left(\frac{4}{3}\right) = 6.66$

So $z = 2x + 3y$ is maximum at $(0,4)$

4. Minimize $z = 2x + y$; subject to the constraints:

$$x + y \geq 3 \quad ; \quad 7x + 5y \leq 35 \quad ; \quad x \geq 0; \; y \geq 0$$

Solution

$x + y \geq 3$ ....(i)

$7x + 5y \leq 35$ ....(ii)

Associated equations

$x + y = 3$ ....(iii)

$7x + 5y = 35$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 3$ then point is $(0,3)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 3$ then point is $(3,0)$

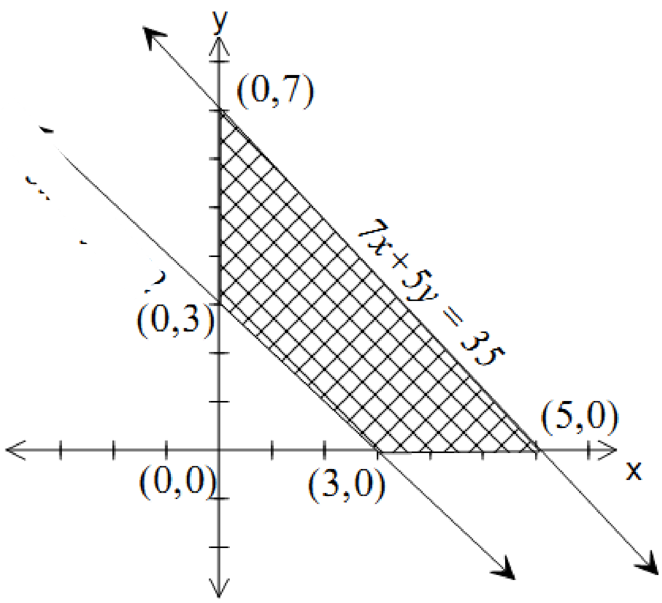
(iv) $\Rightarrow$ Put $x = 0$, $y = 7$ then point is $(0,7)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 5$ then point is $(5,0)$

To check Region put $(\mathbf{0},\mathbf{0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

(i) $\Rightarrow 0 > 3$ false, graph away from the origin

(ii) $\Rightarrow 0 < 35$ true, graph towards the origin



Corner Points: $A(3,0), B(0,3), C(5,0), P(0,7)$

At A: $z = f(3,0) = 2(3) + (0) = 6$

At B: $z = f(0,3) = 2(0) + (3) = 3$

At C: $z = f(5,0) = 2(5) + (0) = 10$

At P: $z = f(0,7) = 2(0) + (7) = 7$

So $z = 2x + y$ is minimum at $(0,3)$

5. Maximize the function defined as; $f(x, y) = 2x + 3y$ subject to the constraints:

$$2x + y \leq 8 \quad ; \quad x + 2y \leq 14 \quad ; \quad x \geq 0; \;\; y \geq 0$$

Solution

$2x + y \leq 8$ ....(i)

$x + 2y \leq 14$ ....(ii)

Associated equations

$2x + y = 8$ ....(iii)

$x + 2y = 14$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 8$ then point is $(0,8)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 4$ then point is $(4,0)$

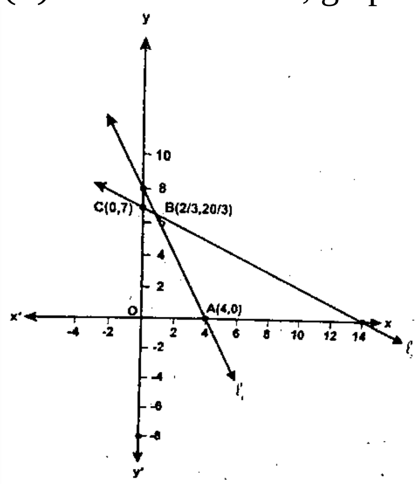
(iv) $\Rightarrow$ Put $x = 0$, $y = 7$ then point is $(0,7)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 14$ then point is $(14,0)$

To check Region put $(0,0)$ in (i) and (ii)

(i) $\Rightarrow 0 < 8$ true, graph towards the origin

(ii) $\Rightarrow 0 < 14$ true, graph towards the origin



Solve 2(iii) − (iv)

$(4x + 2y) - (x + 2y) = 16 - 14$ we have $x = \frac{2}{3}$

Put $x = \frac{2}{3}$ in (iii) we have $y = \frac{20}{3}$ and $C\left(\frac{2}{3}, \frac{20}{3}\right)$

Corner Points: $A(0,0), B(4,0), C\left(\frac{2}{3}, \frac{20}{3}\right), D(0,7)$

At A: $z = f(0,0) = 2(0) + 3(0) = 0$

At B: $z = f(4,0) = 2(4) + 3(0) = 8$

At C: $z = f\left(\frac{2}{3}, \frac{20}{3}\right) = 2\left(\frac{2}{3}\right) + 3\left(\frac{20}{3}\right) = 21.33$

At D: $z = f(0,7) = 2(0) + 3(7) = 21$

So $z = 2x + 3y$ is maximum at $\left(\frac{2}{3}, \frac{20}{3}\right)$

6. Find minimum and maximum values of $z = 3x + y$; subject to the constraints:

$$3x + 5y \geq 15 \quad ; \quad x + 6y \geq 9 \quad ; \quad x \geq 0; \;\; y \geq 0$$

Solution

$3x + 5y \geq 15$ ....(i)

$x + 6y \geq 9$ ....(ii)

Associated equations

$3x + 5y = 15$ ....(iii)

$x + 6y = 9$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 3$ then point is $(0,3)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 5$ then point is $(5,0)$

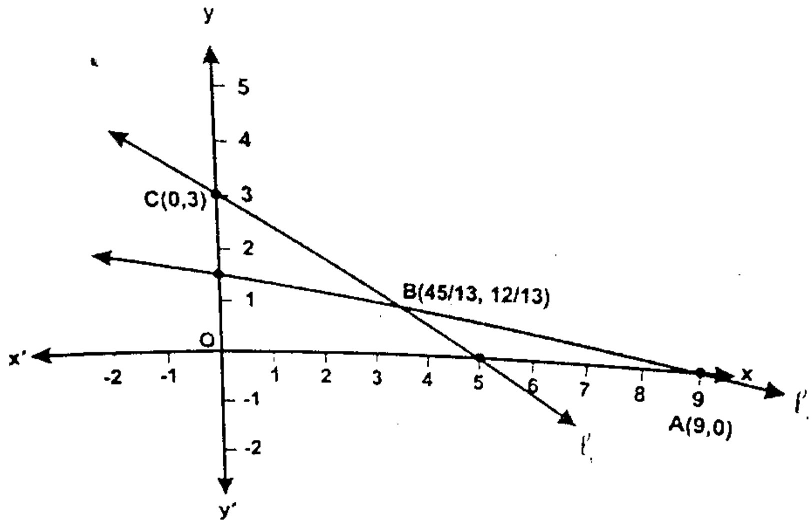
(iv) $\Rightarrow$ Put $x = 0$, $y = \frac{3}{2}$ then point is $\left(0, \frac{3}{2}\right)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 9$ then point is $(9,0)$

To check Region put $(0,0)$ in (i) and (ii)

(i) $\Rightarrow 0 > 15$ false, graph away from the origin

(ii) $\Rightarrow 0 > 9$ false, graph away from the origin



Solve $3(iv) - (iii)$

$(3x + 18y) - (3x + 5y) = 27 - 15$ we have $y = \frac{12}{13}$

Put $y = \frac{12}{13}$ in (iii) we have $x = \frac{45}{13}$ and $B\left(\frac{45}{13}, \frac{12}{13}\right)$

Corner Points: $A(0,3), B\left(\frac{45}{13}, \frac{12}{13}\right), C(9,0)$

At A: $z = f(0,3) = 3(0) + 3 = 3$

At B: $z = f\left(\frac{45}{13}, \frac{12}{13}\right) = 3\left(\frac{45}{13}\right) + \frac{12}{13} = 11.3$

At C: $z = f(9,0) = 3(9) + 0 = 27$

So $z = 3x + y$ is minimum at $(0,3)$ and maximum at $(9,0)$

REVIEW EXERCISE 5

1. Four options are given against each statement. Encircle the correct one.

i. In the following, linear equation is:

(a) $5x > 7$ (b) $4x - 2 < 1$
(c) $2x + 1 = 1$ (d) $4 = 1 + 3$

ii. Solution of $5x - 10 = 10$ is:

(a) 0 (b) 50
(c) 4 (d) -4

iii. If $7x + 4 < 6x + 6$, then x belongs to the interval

(a) $(2, \infty)$ (b) $[2, \infty)$
(c) $(-\infty, 2)$ (d) $(-\infty, 2]$

iv. A vertical line divides the plane into

(a) left half plane (b) right half plane
(c) full plane (d) two half planes

v. The linear equation formed out of the linear inequality is called

(a) linear equation (b) associated equation
(c) quadratic equal (d) none of these

vi. $3x + 4 < 0$ is:

(a) equation (b) inequality
(c) not inequality (d) identity

vii. Corner point is also called:

(a) code (b) vertex
(c) curve (d) region

viii. (0,0) is solution of inequality:

(a) $4x + 5y > 8$ (b) $3x + y > 6$

✓(c) $-2x + 3y < 0$ (d) $x + y > 4$

ix. The solution region restricted to the first quadrant is called:

(a) objective region ✓(b) feasible region

(c) solution region (d) constraints region

x. A function that is to be maximized or minimized is called:

(a) solution function ✓(b) objective function

(c) feasible function (d) none of these

2. Solve and represent their solutions on real line.

(i) $\dfrac{x+5}{3} = 1 - x$ (ii) $\dfrac{2x+1}{3} + \dfrac{1}{2} = 1 - \dfrac{x-1}{3}$

(iii) $3x + 7 < 16$ (iv) $5(x-3) \geq 26x - (10x+4)$

Solution

(i) $\frac{x+5}{3} = 1 - x$

$x + 5 = 3 - 3x$

$x + 3x = 3 - 5$

$4x = -2$

$x = -\frac{2}{4}$

$x = -\frac{1}{2}$

(ii) $\frac{2x+1}{3} + \frac{1}{2} = 1 - \frac{x-1}{3}$

$6 \times \left(\frac{2x+1}{3}\right) + 6 \times \left(\frac{1}{2}\right) = 6 \times (1) - 6 \times \left(\frac{x-1}{3}\right)$

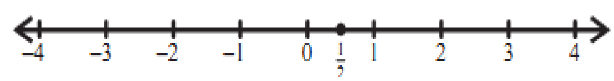
$2(2x+1) + 3 = 6 - 2(x-1)$

$4x + 2 + 3 = 6 - 2x + 2$

$4x + 2x = 6 + 2 - 2 - 3$

$6x = 3$

$x = \frac{1}{2}$



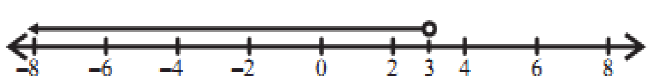
(iii) $3x + 7 < 16$

$3x < 16 - 7$

$3x < 9$

$x < \frac{9}{3}$

$x < 3$



(iv) $5(x-3) \geq 26x - (10x+4)$

$5x - 15 \geq 26x - 10x - 4$

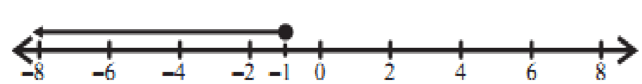
$5x - 15 \geq 16x - 4$

$5x - 16x \geq -4 + 15$

$-11x \geq 11$

$x \leq -\frac{11}{11}$

$x \leq -1$



3. Find the solution region of the following linear equalities:

(i) $3x - 4y \leq 12$; $3x + 2y \geq 3$

(ii) $2x + y \leq 4$; $x + 2y \leq 6$

Solution

3 (i)

$3x - 4y \leq 12$ ....(i)

$3x + 2y \geq 3$ ....(ii)

Associated equations

$3x - 4y = 12$ ....(iii)

$3x + 2y = 3$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = -3$ then point is $(0, -3)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 4$ then point is $(4,0)$

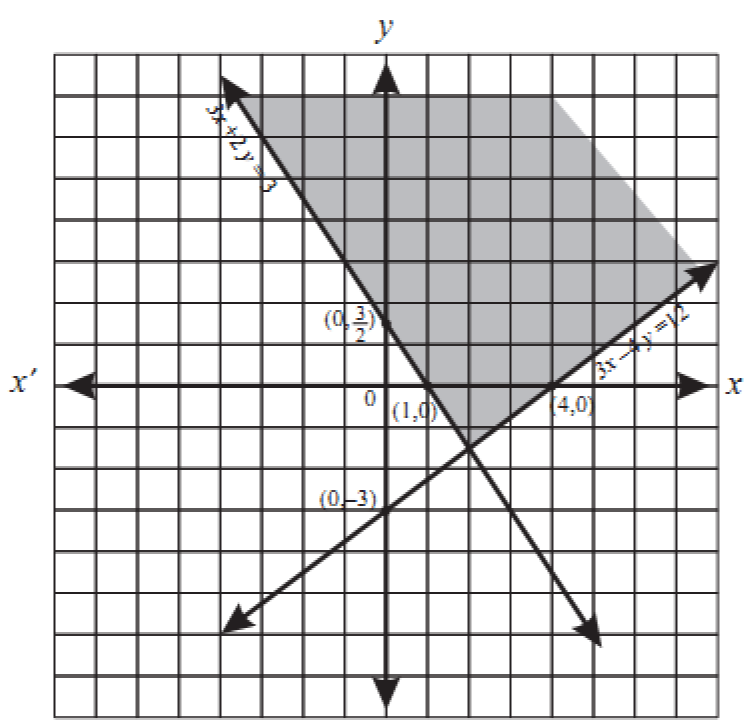
(iv) $\Rightarrow$ Put $x = 0$, $y = \frac{3}{2}$ then point is $\left(0, \frac{3}{2}\right)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 1$ then point is $(1,0)$

To check Region put $(0, 0)$ in (i) and (ii)

(i) $\Rightarrow 0 < 12$ true, graph towards the origin

(ii) $\Rightarrow 0 > 3$ false, graph away from the origin



3 (ii)

$2x + y \leq 4$ ....(i)

$x + 2y \leq 6$ ....(ii)

Associated equations

$2x + y = 4$ ....(iii)

$x + 2y = 6$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 4$ then point is $(0,4)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 2$ then point is $(2,0)$

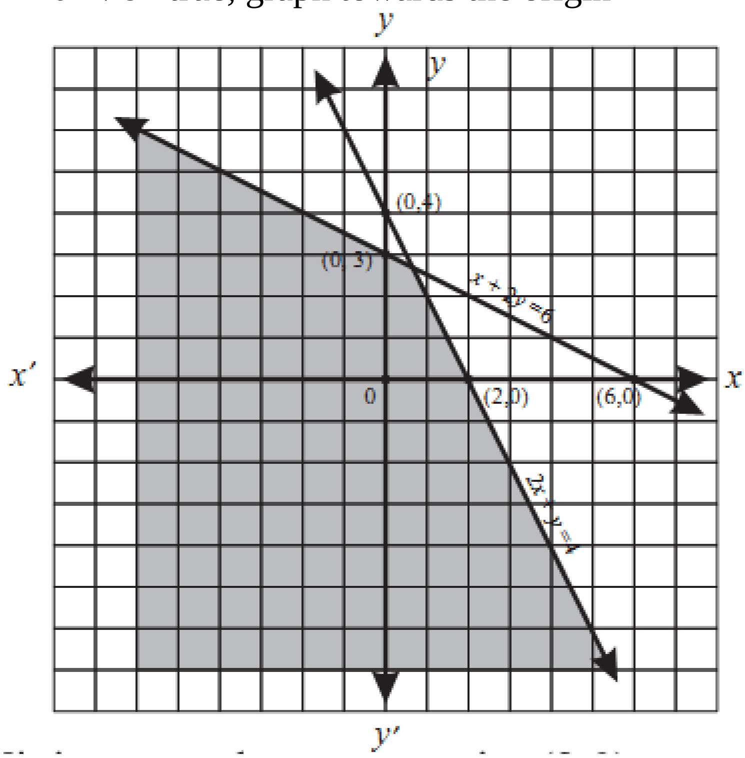
(iv) $\Rightarrow$ Put $x = 0$, $y = 3$ then point is $(0,3)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 6$ then point is $(6,0)$

To check Region put $(0,0)$ in (i) and (ii)

(i) $\Rightarrow 0 < 4$ true, graph towards the origin

(ii) $\Rightarrow 0 < 6$ true, graph towards the origin



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4. Find the maximum value of $g(x,y) = x + 4y$ subject to constraints $x + y \leq 4$, $x \geq 0$ and $y \geq 0$.

Solution

$x + y \leq 4$

Associated equations

$x + y = 4$

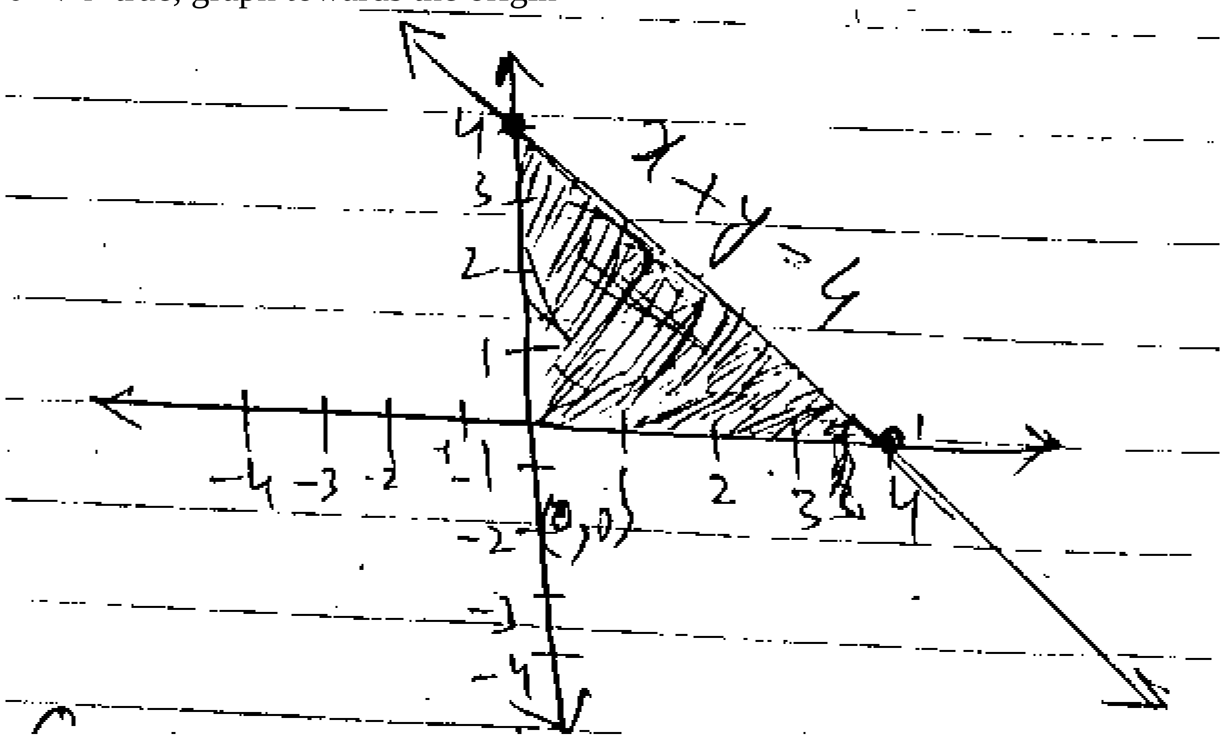
To find Points

$\Rightarrow$ Put $x = 0$, $y = 4$ then point is $(0,4)$

$\Rightarrow$ Put $y = 0$, $x = 4$ then point is $(4,0)$

To check Region put $(\mathbf{0},\mathbf{0})$ in $(\mathbf{i})$ and $(\mathbf{ii})$

$0 < 4$ true, graph towards the origin



Corner Points: $A(0,0), B(0,4), C(4,0)$

At A: $z = g(0,0) = (0) + 4(0) = 0$

At B: $z = g(0,4) = (0) + 4(4) = 16$

At C: $z = g(4,0) = (4) + 0(0) = 4$

So $z = x + 4y$ is maximum at $(0,4)$

5. Find the minimum value of $f(x,y) = 3x + 5y$ subject to constraints
$x + 3y \geq 3, \quad x + y \geq 2, \quad x \geq 0, \quad y \geq 0.$

Solution

$x + 3y \geq 3$ ....(i)

$x + y \geq 2$ ....(ii)

Associated equations

$x + 3y = 3$ ....(iii)

$x + y = 2$ ....(iv)

To find Points

(iii) $\Rightarrow$ Put $x = 0$, $y = 1$ then point is $(0,1)$

(iii) $\Rightarrow$ Put $y = 0$, $x = 3$ then point is $(3,0)$

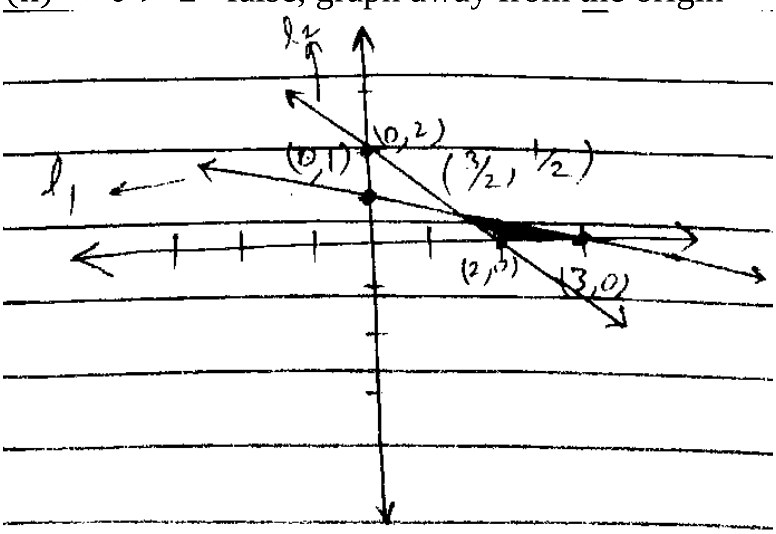
(iv) $\Rightarrow$ Put $x = 0$, $y = 2$ then point is $(0,2)$

(iv) $\Rightarrow$ Put $y = 0$, $x = 2$ then point is $(2,0)$

To check Region put $(0,0)$ in (i) and (ii)

(i) $\Rightarrow 0 > 3$ false, graph away from the origin

(ii) $\Rightarrow 0 > 2$ false, graph away from the origin



Solve (iii) − (iv)

$(x + 3y) - (x + y) = 3 - 2$ we have $y = \frac{1}{2}$

Put $y = \frac{1}{2}$ in (iii) we have $x = \frac{3}{2}$ and $P\left(\frac{3}{2}, \frac{1}{2}\right)$

Corner Points: $A(2,0), B(3,0), P\left(\frac{3}{2}, \frac{1}{2}\right)$

At A: $z = f(2,0) = 3(2) + 5(0) = 6$

At B: $z = f(3,0) = 3(3) + 5(0) = 9$

At P: $z = f\left(\frac{3}{2}, \frac{1}{2}\right) = 3\left(\frac{3}{2}\right) + 5\left(\frac{1}{2}\right) = 8$

So $z = 3x + 5y$ is minimum at $(2,0)$ and maximum at $(3,0)$

Unit 6 Trigonometry

EXERCISE 6.1

1. Find in which quadrant the following angles lie. Write a co-terminal angle for each:

(i) $65°$ (ii) $135°$ (iii) $-40°$ (iv) $210°$ (v) $-150°$

Solution

(i) 1^{st} (ii) 2^{nd} (iii) 4^{th} (iv) 3^{rd} (v) 3^{rd}

2. Convert the following into degrees, minutes, and seconds:

(i) $123.456°$ (ii) $58.7891°$ (iii) $90.5678°$

Solution

2(i): $\mathbf{123.456°}$

123

$0.456 \times 60 = 27.36$

$0.36 \times 60 = 21.6$

$123.456° \approx 123°\, 27'\, 22''$

2(ii): $\mathbf{58.7891°}$

58

$0.7891 \times 60 = 47.346$

$0.346 \times 60 = 20.76$

$58.7891° \approx 58°\, 47'\, 21''$

2(iii): $\mathbf{90.5678°}$

90

$0.5678 \times 60 = 34.068$

$0.068 \times 60 = 4.08$

$90.5678° \approx 90°\, 34' 4''$

3. Convert the following into decimal degrees:

(i) 65° 32' 15" (ii) 42° 18' 45" (iii) 78° 45' 36"

Solution

3(i): $65°32'15''$

$65°32'15'' = 65 + \frac{32}{60} + \frac{15}{60\times60} = 65 + 0.5333 + 0.0042 = 65.5375°$

3(ii): $42°18'45''$

$42°18'45'' = 42 + \frac{18}{60} + \frac{45}{60\times60} = 42 + 0.3 + 0.0125 = 42.3125°$

3(iii): $78°45'36''$

$78°45'36'' = 78 + \frac{45}{60} + \frac{36}{60\times60} = 78 + 0.75 + 0.01 = 78.76°$

4. Convert the following into radians:

(i) 36° (ii) 22.5° (iii) 67.5°

Solution

4(i): $\mathbf{36°} = 36 \times \frac{\pi}{180} = \frac{\pi}{5}\text{rad}$

4(ii):$\mathbf{22.5°} = 22.5 \times \frac{\pi}{180} = \frac{\pi}{8}\text{rad}$

4(iii):$\mathbf{67.5°} = 67.5 \times \frac{\pi}{180} = \frac{3\pi}{8}\text{rad}$

5. Convert the following into degrees:

(i) $\frac{\pi}{16}$rad (ii) $\frac{11\pi}{5}$rad (iii) $\frac{7\pi}{6}$rad

Solution

5(i): $\frac{\boldsymbol{\pi}}{\mathbf{16}}\textbf{rad} = \frac{\pi}{16} \times \frac{180°}{\pi} = 11.25°$

5(ii): $\frac{\mathbf{11\pi}}{\mathbf{5}}\textbf{rad} = \frac{11\pi}{5} \times \frac{180°}{\pi} = 396°$

5(iii): $\frac{\mathbf{7\pi}}{\mathbf{6}}\textbf{rad} = \frac{7\pi}{6} \times \frac{180°}{\pi} = 210°$

6. Find the arc length and area of a sector if:

(i) $r = 6$ cm and central angle $\theta = \frac{\pi}{3}$ radians.

(ii) $r = \frac{4.8}{\pi}$ cm and central angle $\theta = \frac{5\pi}{6}$ radians.

Solution

6(i): $\boldsymbol{l} = r\theta = 6 \times \frac{\pi}{3} = 6.28\text{cm}$

$\boldsymbol{A} = \frac{1}{2}r^2\theta = \frac{1}{2} \times (6)^2 \times \frac{\pi}{3} = 18.84\text{cm}^2$

6(ii): $\boldsymbol{l} = r\theta = \frac{4.8}{\pi} \times \frac{5\pi}{6} = 4\text{cm}$

$\boldsymbol{A} = \frac{1}{2}r^2\theta = \frac{1}{2} \times \left(\frac{4.8}{\pi}\right)^2 \times \frac{5\pi}{6} = 3.06\text{cm}^2$

7. If the central angle of a sector is 60° and the radius of the circle is 12 cm, find the area of the sector and the percentage of the total area of the circle it represents.

Solution

$\theta = 60° = 60 \times \frac{\pi}{180} = \frac{\pi}{3}\text{rad}$

Area of the sector $= \frac{1}{2}r^2\theta = \frac{1}{2} \times (12)^2 \times \frac{\pi}{3} = 62.83\text{cm}^2$

Total area of the circle $= \pi r^2 = 3.14159 \times (12)^2 = 452.389\text{cm}^2$

Percentage $= \frac{\text{Area of the sector}}{\text{Total area of the circle}} \times 100\%$

Percentage $= \frac{62.83\text{cm}^2}{452.389\text{cm}^2} \times 100\% = 13.89\%$

8. Find the percentage of the area of sector subtending an angle $\frac{\pi}{8}$ radians.

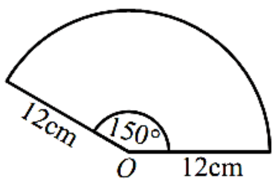
Solution

Percentage $= \frac{\text{Area of the sector}}{\text{Total area of the circle}} \times 100\%$

Percentage $= \frac{\theta}{2\pi} \times 100\% = \frac{\frac{\pi}{8}}{2\pi} \times 100\% = 6.25\%$

9. A circular sector of radius $r = 12$ cm has an angle of 150°. This sector is cut out and then bent to form a cone. What is the slant height and the radius of the base of this cone?

Hint: Arc length of sector = circumference of cone.



Solution

Radius of the sector $= r = 12\text{cm}$

Angle of the sector $= \theta = 150° = 150 \times \frac{\pi}{180} = \frac{5\pi}{6}\text{rad}$

Arc Length $= l = r\theta = 12 \times \frac{5\pi}{6} = 10\pi\text{cm}$

Now

Circumference of base of the cone $= 2\pi r'$

$10\pi = 2\pi r'$

radius of base $= r' = 5\text{cm}$

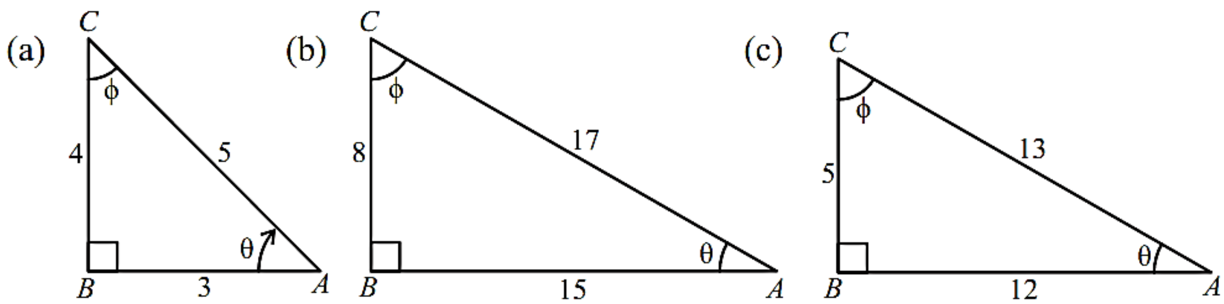
slant height $= l = r = 12\text{cm}$

EXERCISE 6.2

1. For each of the following right-angled triangles, find the trigonometric ratios:

(i) $\sin\theta$ (ii) $\cos\theta$ (iii) $\tan\theta$ (iv) $\sec\theta$ (v) $\operatorname{cosec}\theta$

(vi) $\cot\phi$ (vii) $\tan\phi$ (viii) $\operatorname{cosec}\phi$ (ix) $\sec\phi$ (x) $\cos\phi$



Solution

(a) (i) $\frac{4}{5}$ (ii) $\frac{3}{5}$ (iii) $\frac{4}{3}$ (iv) $\frac{5}{3}$ (v) $\frac{5}{4}$ (vi) $\frac{4}{3}$ (vii) $\frac{3}{4}$ (viii) $\frac{5}{3}$ (ix) $\frac{5}{4}$ (x) $\frac{4}{5}$

(b) (i) $\frac{8}{17}$ (ii) $\frac{15}{17}$ (iii) $\frac{8}{15}$ (iv) $\frac{17}{15}$ (v) $\frac{17}{8}$ (vi) $\frac{8}{15}$ (vii) $\frac{15}{8}$ (viii) $\frac{17}{15}$ (ix) $\frac{17}{8}$ (x) $\frac{8}{17}$

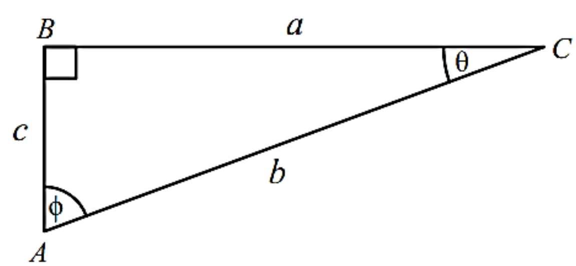
(c) (i) $\frac{5}{13}$ (ii) $\frac{12}{13}$ (iii) $\frac{5}{12}$ (iv) $\frac{13}{5}$ (v) $\frac{13}{12}$ (vi) $\frac{5}{12}$ (vii) $\frac{12}{5}$ (viii) $\frac{13}{12}$ (ix) $\frac{13}{5}$ (x) $\frac{5}{13}$

2. For the following right-angled triangle ABC find the trigonometric ratios for which $m\angle A = \phi$ and $m\angle C = \theta$

(i) $\sin\theta$ (ii) $\cos\theta$

(iii) $\tan\theta$ (iv) $\sin\phi$

(v) $\cos\phi$ (vi) $\tan\phi$



Solution

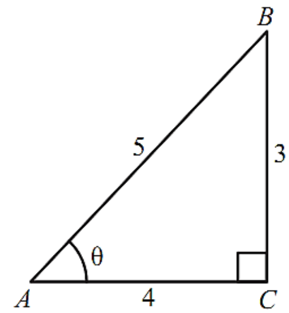
(i) $\frac{c}{b}$ (ii) $\frac{a}{b}$ (iii) $\frac{c}{a}$ (iv) $\frac{a}{b}$ (v) $\frac{c}{b}$ (vi) $\frac{a}{c}$

3. Considering the adjoining triangle ABC, verify that:

(i) $\sin\theta \ \text{cosec}\,\theta = 1$

(ii) $\cos\theta \ \sec\theta = 1$

(iii) $\tan\theta \ \cot\theta = 1$



Solution

3.(i) $sin\theta cosec\theta = \frac{3}{5} \times \frac{5}{4} = 1$

3.(ii) $cos\theta sec\theta = \frac{4}{5} \times \frac{5}{4} = 1$

3.(iii) $tan\theta cot\theta = \frac{3}{4} \times \frac{4}{3} = 1$

4. Fill in the blanks.

(i) $\sin 30° = \sin(90° - 60°) =$ $cos60°$

(ii) $\cos 30° = \cos(90° - 60°) =$ $sin60°$

(iii) $\tan 30° = \tan(90° - 60°) =$ $cot60°$

(iv) $\tan 60° = \tan(90° - 30°) =$ $cot30°$

(v) $\sin 60° = \sin(90° - 30°) =$ $cos30°$

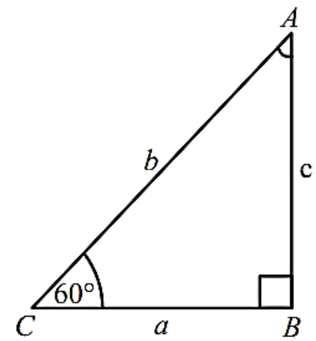
(vi) $\cos 60° = \cos(90° - 30°) =$ $sin30°$

(vii) $\sin 45° = \sin(90° - 45°) =$ $cos45°$

(viii) $\tan 45° = \tan(90° - 45°) =$ $cot45°$

(ix) $\cos 45° = \cos(90° - 45°) =$ $sin45°$

5. In a right angled triangle ABC, $m\angle B = 90°$ and C is an acute angle of $60°$. Also $\sin m\angle A = \frac{a}{b}$, then find the following trigonometric ratios:



(i) $\frac{m\overline{BC}}{m\overline{AB}}$ (ii) $\cos 60°$

(iii) $\tan 60^\circ$ (iv) $\operatorname{cosec} \frac{\pi}{3}$

(v) $\cot 60°$ (vi) $\sin 30°$

(vii) $\cos 30°$ (viii) $\tan \frac{\pi}{6}$

(ix) $\sec 30°$ (x) $\cot 30°$

Solution

(i) $\frac{a}{b}$ (ii) $\frac{a}{b}$ (iii) $\frac{c}{a}$ (iv) $\frac{b}{c}$ (v) $\frac{a}{c}$

(vi) $\frac{a}{b}$ (vii) $\frac{c}{b}$ (viii) $\frac{a}{c}$ (ix) $\frac{b}{c}$ (x) $\frac{c}{a}$

EXERCISE 6.3

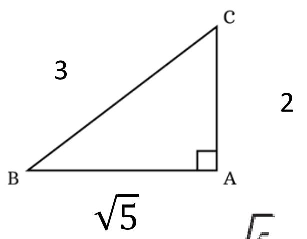
1. If θ lies in first quadrant, find the remaining trigonometric ratios of θ.

(i) $\sin\theta = \frac{2}{3}$ (ii) $\cos\theta = \frac{3}{4}$ (iii) $\tan\theta = \frac{1}{2}$

(iv) $\sec\theta = 3$ (v) $\cot\theta = \sqrt{\frac{3}{2}}$

Solution

1.(i) $\boldsymbol{\sin\theta = \frac{2}{3}}$



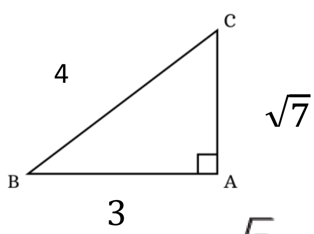
By Pythagoras Formula

$$H^2 = P^2 + B^2 \Rightarrow 3^2 = 2^2 + B^2$$

$$\Rightarrow B^2 = 9 - 4 = 5 \Rightarrow B = \sqrt{5}$$

(i) $\cos\theta = \frac{\sqrt{5}}{3}$, $\tan\theta = \frac{2}{\sqrt{5}}$, $\operatorname{cosec}\theta = \frac{3}{2}$, $\sec\theta = \frac{3}{\sqrt{5}}$, $\cot\theta = \frac{\sqrt{5}}{2}$

1.(ii) $\boldsymbol{\cos\theta = \frac{3}{4}}$



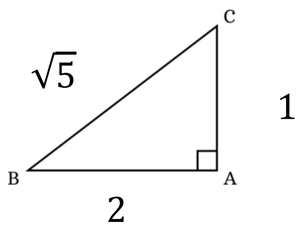
By Pythagoras Formula

$$H^2 = P^2 + B^2 \Rightarrow 4^2 = P^2 + 3^2$$

$$\Rightarrow P^2 = 16 - 9 = 7 \Rightarrow P = \sqrt{7}$$

(ii) $\sin\theta = \frac{\sqrt{7}}{4}$, $\tan\theta = \frac{\sqrt{7}}{3}$, $\operatorname{cosec}\theta = \frac{4}{\sqrt{7}}$, $\sec\theta = \frac{4}{3}$, $\cot\theta = \frac{3}{\sqrt{7}}$

1.(iii) $\boldsymbol{\tan\theta = \frac{1}{2}}$



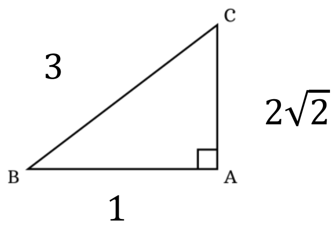
By Pythagoras Formula

$$H^2 = P^2 + B^2 \Rightarrow H^2 = 1^2 + 2^2$$

$$\Rightarrow H^2 = 1 + 4 = 5 \Rightarrow H = \sqrt{5}$$

(iii) $\sin\theta = \frac{1}{\sqrt{5}}$, $\cos\theta = \frac{2}{\sqrt{5}}$, $\operatorname{cosec}\theta = \sqrt{5}$, $\sec\theta = \frac{\sqrt{5}}{2}$, $\cot\theta = 2$

1.(iv) $\boldsymbol{sec\theta = 3 = \frac{3}{1}}$



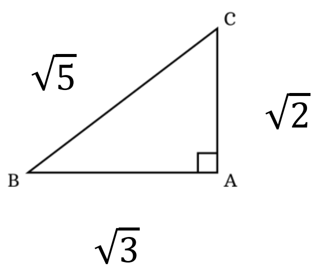
By Pythagoras Formula

$$H^2 = P^2 + B^2 \Rightarrow 3^2 = P^2 + 1^2$$

$$\Rightarrow P^2 = 9 - 1 = 8 \Rightarrow P = 2\sqrt{2}$$

(iv) $\sin\theta = \frac{2\sqrt{2}}{3}$, $\cos\theta = \frac{1}{3}$, $\tan\theta = 2\sqrt{2}$, $\text{cosec}\,\theta = \frac{3}{2\sqrt{2}}$, $\cot\theta = \frac{1}{2\sqrt{2}}$

1.(v) $\boldsymbol{cot\theta = \sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}}}$



By Pythagoras Formula

$$H^2 = P^2 + B^2 \Rightarrow H^2 = \left(\sqrt{2}\right)^2 + \left(\sqrt{3}\right)^2$$

$$\Rightarrow H^2 = 2 + 3 = 5 \Rightarrow H = \sqrt{5}$$

(v) $\sin\theta = \sqrt{\frac{2}{5}}$, $\cos\theta = \sqrt{\frac{3}{5}}$, $\tan\theta = \sqrt{\frac{2}{3}}$, $\text{cosec}\,\theta = \sqrt{\frac{5}{2}}$, $\sec\theta = \sqrt{\frac{5}{3}}$

Prove the Following Trigonometric Identities

2. $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

Solution

$(\sin\theta + \cos\theta)^2 = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$

$(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$

3. $\dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$

Solution

$\frac{\cos\theta}{\sin\theta} = \cot\theta = \frac{1}{\tan\theta}$

4. $$\frac{\sin\theta}{\operatorname{cosec}\theta}+\frac{\cos\theta}{\sec\theta}=1$$

Solution

$$\frac{\sin\theta}{\operatorname{cosec}\theta}+\frac{\cos\theta}{\sec\theta}=\sin\theta\times\frac{1}{\operatorname{cosec}\theta}+\cos\theta\times\frac{1}{\sec\theta}$$
$$\frac{\sin\theta}{\operatorname{cosec}\theta}+\frac{\cos\theta}{\sec\theta}=\sin\theta\times\sin\theta+\cos\theta\times\cos\theta=\sin^2\theta+\cos^2\theta=1$$

5. $$\cos^2\theta-\sin^2\theta=2\cos^2\theta-1$$

Solution

$$\cos^2\theta-\sin^2\theta=\cos^2\theta-(1-\cos^2\theta)=\cos^2\theta-1+\cos^2\theta$$
$$\cos^2\theta-\sin^2\theta=2\cos^2\theta-1$$

6. $$\cos^2\theta-\sin^2\theta=1-2\sin^2\theta$$

Solution

$$\cos^2\theta-\sin^2\theta=(1-\sin^2\theta)-\sin^2\theta=1-\sin^2\theta-\sin^2\theta$$
$$\cos^2\theta-\sin^2\theta=1-2\sin^2\theta$$

7. $$\frac{1-\sin\theta}{\cos\theta}=\frac{\cos\theta}{1+\sin\theta}$$

Solution

$$\frac{1-\sin\theta}{\cos\theta}=\frac{(1-\sin\theta)(1+\sin\theta)}{\cos\theta(1+\sin\theta)}=\frac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)}=\frac{\cos^2\theta}{\cos\theta(1+\sin\theta)}=\frac{\cos\theta}{1+\sin\theta}$$

8. $$(\sec\theta-\tan\theta)^2=\frac{1-\sin\theta}{1+\sin\theta}$$

Solution

$$(\sec\theta-\tan\theta)^2=\left(\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\right)^2=\left(\frac{1-\sin\theta}{\cos\theta}\right)^2=\frac{(1-\sin\theta)^2}{\cos^2\theta}=\frac{(1-\sin\theta)(1-\sin\theta)}{1-\sin^2\theta}$$
$$(\sec\theta-\tan\theta)^2=\frac{(1-\sin\theta)(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}=\frac{1-\sin\theta}{1+\sin\theta}$$

9. $$(\tan\theta+\cot\theta)^2=\sec^2\theta\operatorname{cosec}^2\theta$$

Solution

$$(\tan\theta+\cot\theta)^2=\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)^2=\left(\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right)^2=\left(\frac{1}{\cos\theta\sin\theta}\right)^2$$
$$(\tan\theta+\cot\theta)^2=\frac{1}{\cos^2\theta}\times\frac{1}{\sin^2\theta}=\sec^2\theta\operatorname{cosec}^2\theta$$

$$10. \quad \frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\tan\theta+\sec\theta$$

Solution

$\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}$

$=\frac{\tan\theta+\sec\theta-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1}$

$=\frac{\tan\theta+\sec\theta-(\sec\theta-\tan\theta)(\sec\theta+\tan\theta)}{\tan\theta-\sec\theta+1}$

$=\frac{(\tan\theta+\sec\theta)[1-(\sec\theta-\tan\theta)]}{\tan\theta-\sec\theta+1}$

$=\frac{(\tan\theta+\sec\theta)[1-\sec\theta+\tan\theta]}{1-\sec\theta+\tan\theta}$

$=\tan\theta+\sec\theta$

$$11. \quad \sin^3\theta-\cos^3\theta=(\sin\theta-\cos\theta)(1+\sin\theta\cos\theta)$$

Solution

$\sin^3\theta-\cos^3\theta$

$=(\sin\theta-\cos\theta)(\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta)$

$=(\sin\theta-\cos\theta)(1+\sin\theta\cos\theta)$

$$12. \quad \sin^6\theta-\cos^6\theta=(\sin^2\theta-\cos^2\theta)(1-\sin^2\theta\cos^2\theta)$$

Solution

$\sin^6\theta-\cos^6\theta$

$=(\sin^2\theta)^3-(\cos^2\theta)^3$

$=(\sin^2\theta-\cos^2\theta)[(\sin^2\theta)^2+(\cos^2\theta)^2+(\sin^2\theta\cos^2\theta)]$

$=(\sin^2\theta-\cos^2\theta)[(\sin^2\theta)^2+(\cos^2\theta)^2+2\sin^2\theta\cos^2\theta-\sin^2\theta\cos^2\theta]$

$=(\sin^2\theta-\cos^2\theta)[(\sin^2\theta+\cos^2\theta)^2-\sin^2\theta\cos^2\theta]$

$=(\sin^2\theta-\cos^2\theta)(1-\sin^2\theta\cos^2\theta)$

EXERCISE 6.4

θ	0	$30° = \frac{\pi}{6}$	$45° = \frac{\pi}{4}$	$60° = \frac{\pi}{3}$	$90° = \frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

1. Find the value of the following trigonometric ratios without using the calculator.

(i) sim 30° (ii) cos 30° (iii) $\tan\frac{\pi}{6}$ (iv) tan 60°

(v) sec 60° (vi) $\cos\frac{\pi}{3}$ (vii) cot 60° (viii) sin 60°

(ix) sec 30° (x) cosec 30° (xi) sin 45° (xii) $\cos\frac{\pi}{4}$

Solution

(i) $\frac{1}{2}$ (ii) $\frac{\sqrt{3}}{2}$ (iii) $\frac{\sqrt{3}}{3}$ (iv) $\sqrt{3}$

(v) 2 (vi) $\frac{1}{2}$ (vii) $\frac{\sqrt{3}}{3}$ (viii) $\frac{\sqrt{3}}{2}$

(ix) $\frac{2\sqrt{3}}{3}$ (x) 2 (xi) $\frac{\sqrt{2}}{2}$ (xii) $\frac{\sqrt{2}}{2}$

2. Evaluate:

(i) $2\sin 60^\circ \cos 60^\circ$ (ii) $2\cos\dfrac{\pi}{3}\sin\dfrac{\pi}{3}$

(iii) $2\sin 45^\circ + 2\cos 45^\circ$ (iv) $\sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$

(v) $\cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ$ (vi) $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$

(vii) $\cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ$ (viii) $\tan\dfrac{\pi}{6}\cot\dfrac{\pi}{6} + 1$

Solution

2(i): $2\sin60^\circ\cos60^\circ = 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{2}$

2(ii): $2\cos\frac{\pi}{3}\sin\frac{\pi}{3} = 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

2(iii): $2\sin45^\circ + 2\cos45^\circ = 2 \times \frac{1}{\sqrt{2}} + 2 \times \frac{1}{\sqrt{2}} = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

2(iv): $\sin60^\circ\cos30^\circ + \cos60^\circ\sin30^\circ = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$

2(v): $\cos60^\circ\cos30^\circ - \sin60^\circ\sin30^\circ = \frac{1}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$

2(vi): $\sin60^\circ\cos30^\circ - \cos60^\circ\sin30^\circ = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

2(vii): $\cos60^\circ\cos30^\circ + \sin60^\circ\sin30^\circ = \frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

2(viii): $\tan\frac{\pi}{6}\cot\frac{\pi}{6} + 1 = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{1} + 1 = 1 + 1 = 2$

3. If $\sin\dfrac{\pi}{4}$ and $\cos\dfrac{\pi}{4}$ equal to $\dfrac{1}{\sqrt{2}}$ each, then find the value of the followings:

(i) $2\sin 45^\circ - 2\cos 45^\circ$ (ii) $3\cos 45^\circ + 4\sin 45^\circ$

(iii) $5\cos 45^\circ - 3\sin 45^\circ$

Solution

3(i): $2\sin45^\circ - 2\cos45^\circ = 2 \times \frac{1}{\sqrt{2}} - 2 \times \frac{1}{\sqrt{2}} = \sqrt{2} - \sqrt{2} = 0$

3(ii): $3\cos45^\circ + 4\sin45^\circ = 3 \times \frac{1}{\sqrt{2}} + 4 \times \frac{1}{\sqrt{2}} = \frac{7}{\sqrt{2}}$

3(iii): $5\cos45^\circ - 3\sin45^\circ = 5 \times \frac{1}{\sqrt{2}} - 3 \times \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \frac{\sqrt{2}\times\sqrt{2}}{\sqrt{2}} = \sqrt{2}$